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PROJECTIVE MODULES IN THE CATEGORY $\mathcal{O}_S$: SELF-DUALITY

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ABSTRACT. Given a parabolic subalgebra $\mathfrak{p}_S$ of a complex, semisimple Lie algebra $\mathfrak{g}$, there is a naturally defined category $\mathcal{O}_S$ of $\mathfrak{g}$-modules which includes all the $\mathfrak{g}$-modules induced from finite-dimensional $\mathfrak{p}_S$-modules. This paper treats the question of whether certain projective modules in $\mathcal{O}_S$ are isomorphic to their dual modules. The projectives in question are the projective covers of those simple modules occurring in the socles of generalized Verma modules. Their self-duality is proved in a number of cases, and additional information is obtained on the generalized Verma modules.

1. Introduction. In this paper we study the self-duality of projective modules within certain module categories over semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let $\mathfrak{p}_S$ be a parabolic subalgebra corresponding to a subset S of simple roots, and choose a Cartan subalgebra $\mathfrak{h}_S$ of the semisimple part of $\mathfrak{p}_S$ (see §2 for a precise description of these objects). Associated to $\mathfrak{p}_S$ is a category $\mathcal{O}_S$ of modules, which coincides with the usual category $\mathcal{O}$ in case $\mathfrak{p} = \mathfrak{b}$. All the modules have finite length, and the composition factors are simple highest weight modules $L(\mu)$, with the highest weight μ restricted to be dominant integral on $\mathfrak{h}_S$. Associated to each such weight μ is the generalized Verma module $M_S(\mu)$, which is a universal highest weight module in $\mathcal{O}_S$, and a projective cover $P_S(\mu)$ of $L(\mu)$, these projectives exhausting the indecomposable projectives of $\mathcal{O}_S$. Each $P_S(\mu)$ has a finite filtration with generalized Verma modules as successive quotients (we will call such a filtration a Verma flag), and there is a duality functor D on $\mathcal{O}_S$ which fixes the simple modules. If a projective $P_S(\mu)$ is isomorphic to its dual $DP_S(\mu)$, then $P_S(\mu)$ is the injective envelope in $\mathcal{O}_S$ of $L(\mu)$, which is its socle. As a consequence, $L(\mu)$ must be the socle of the generalized Verma module occurring at the bottom of a Verma flag of $P_S(\mu)$, which yields a necessary condition on $L(\mu)$ if $P_S(\mu)$ is to be self-dual. The primary question treated in this paper is the converse:

CONJECTURE. *Let μ be a weight in $\mathfrak{h}^*$ which is dominant integral on $\mathfrak{h}_S^*$. If $L(\mu)$ is a summand of the socle of some generalized Verma module in $\mathcal{O}_S$, then $P_S(\mu)$ and $DP_S(\mu)$ are isomorphic.*

For convenience, we will call $L(\mu)$, or μ, *socular* if $L(\mu)$ is a summand of the socle of a generalized Verma module. The socular simple modules are characterized in §4.3 via several equivalent conditions. For instance, in a given block of $\mathcal{O}_S$, they are the simple modules of largest Gelfand-Kirillov dimension, an observation of D.

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Vogan. These characterizations could conceivably be a starting point for a general proof of the Conjecture[1]. In particular, they imply that the Conjecture is true for a given block if it is true for a single socular $L(\mu)$ in the block (§9.2). However, we will not make use of this before §9. Instead, we will prove the Conjecture in a number of special cases, via a procedure sketched below which produces the socular simples along the way, independently of the characterizations.

In case $\mathfrak{p} = \mathfrak{b}$, it is a well-known result of Borho that for any weight μ, the socle of $M(\mu)$ is the simple module $L(\nu)$ corresponding to the unique antidominant weight ν in $W_\mu \cdot \mu$. The Conjecture reduces to the statement that $P(\nu)$ is self-dual for any antidominant weight ν. In case ν is integral, this was essentially proved by Humphreys via an elementary argument [**7**, see §9.1], and for arbitrary antidominant ν the first proof was given in [**8**, 5.3]. A short, elegant proof using a theorem of Bernstein and Gelfand has been given by Joseph [**11**, 3.13], and in §3 we provide a third proof streamlined from the ideas of §4.

Given a regular weight μ which is dominant integral on $\mathfrak{h}_S^*$, we show in §4 that if μ satisfies a condition we call specialness, then $P_S(\mu)$ is self-dual. Moreover, the self-duality of $P_S(\mu)$ implies the self-duality of $P_S(\eta)$ for any $\eta \ll \mu$, where $\eta \ll \mu$ means there is a sequence of *simple* roots $\alpha_1, \ldots, \alpha_r$ such that $\eta = s_{\alpha_1} \cdots s_{\alpha_r} \cdot \mu < s_{\alpha_2} \cdots s_{\alpha_r} \cdot \mu < \cdots < s_{\alpha_r} \cdot \mu < \mu$. Thus, the Conjecture would follow if all socular weights which are maximal with respect to $\ll$ are special. This can be proved in many special cases via a procedure outlined in §4.7. First one determines the composition factors of the generalized Verma modules in the block of $\mathcal{O}_S$ under consideration. From this, one guesses that a set X of weights is the set of socular weights, and verifies that the maximal elements of X are special. This yields a supply $\{P_S(\mu) | \mu \in X\}$ of self-dual projectives, to each of which corresponds a generalized Verma module $M_S(\check{\mu})$ with socle $L(\mu)$ (see §4.6). By examining $M_S(\eta)$ for η not equal to any $\check{\mu}$ with $\mu \in X$, one shows that X is the complete set of socular weights in the block, verifying the Conjecture. We carry out this procedure in §§5–8 for a number of cases. In §9, we discuss nonregular weights and use them to obtain the Conjecture in additional cases.

In [**5**], Enright and Shelton study the class $\mathcal{D}$ of self-dual modules in a block $\mathcal{B}$ of $\mathcal{O}_S$ which have a Verma flag. They assume first that every generalized Verma module $M_S(\mu)$ in $\mathcal{B}$ has a simple socle $L(\mu^\#)$, and associate to μ the homomorphic image $D_S(\mu)$ of $P_S(\mu^\#)$ obtained by omitting from the Verma flag of $P_S(\mu^\#)$ all constituents of the form $M_S(\eta)$ with $\eta \not< \mu$. Under the second assumption on $\mathcal{B}$ that each $D_S(\mu)$ is self-dual, they prove a number of results, including the fact that every module in $\mathcal{D}$ is a direct sum of $D_S(\mu)$'s. In case $\mathfrak{g}$ is of type $A - D$ and $\mathfrak{p}_S$ is the largest possible parabolic, they verify the two assumptions, and in these cases the Conjecture follows. In fact, their consideration of the D_n case led me to extend my earlier results on the A_n case to D_n, and to formulate the Conjecture. Moreover, the procedure of §4.7 described above leads to an alternate verification of their two assumptions in these cases, as well as in the cases where $\mathfrak{g}$ is of type E_6 (or E_7), with $\mathfrak{p}_S$ of type D_5 (or E_6). The first assumption, on simplicity of the socle, is not true in general (for instance, see [**2**]) and fails in particular for $(\mathfrak{g}, \mathfrak{p}_S)$ of type (F_4, C_3). Also, it turns out that if the first assumption is true, so that the

[1]Indeed, this is the case. A general proof is given in the addendum.

modules $D_S(\mu)$ are defined, the second assumption may still be false, as one can see in several examples of this paper (see §7.3).

NOTES. (1) In a successor to this paper, I will study the structure of the self-dual projective modules in greater detail, analyzing their Loewy length and Loewy series in terms of the Loewy data for generalized Verma modules.

(2) Some of the results of this paper on generalized Verma modules overlap with results of D. H. Collingwood, and of B. D. Boe and Collingwood (see [**2** and **18**], among other papers). For the most part, I have minimized the overlap by simply referring to their more complete results on composition factors. But in addition, some of the structural information implicit in this paper (to be made explicit in the successor) can be gleaned from their papers.

(3) I wish to thank T. Enright and B. Shelton for conversations on the material of this paper, and D. Vogan for his remarks contained in §4.3.

(4) This paper is dedicated to my mother on the occasion of her 65th birthday.

2. Notation and background.

2.0 The notation is essentially that used by Jantzen in [**10**], with occasional modifications, and any unexplained notions can be found there. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and corresponding root system R. The transpose anti-automorphism t is defined on $\mathfrak{g}$ with reference to a fixed Chevalley basis, fixing $\mathfrak{h}$ and switching root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for $\alpha \in R$. Choose a base B in R of simple roots, corresponding to which are defined the positive roots R^+, the Borel subalgebra $\mathfrak{b}$, and its nilradical $\mathfrak{n}^+$.

For a subset S of B, define $R_S = R \cap \mathbf{Z}S$ and $\mathfrak{h}_S = \sum_{\alpha \in S}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Let $\mathfrak{g}_S = \mathfrak{h}_S \oplus \mathfrak{n}_S \oplus \mathfrak{n}_S^-$, where $\mathfrak{n}_S = \bigoplus_{\alpha \in R_S \cap R^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_S^- = {}^t\mathfrak{n}_S$. Set $\mathfrak{m}_S = \bigoplus_{\alpha \in R^+ \setminus R_S} \mathfrak{g}_\alpha$ and $\mathfrak{p}_S = (\mathfrak{g}_S + \mathfrak{h}) + \mathfrak{m}_S$, so that $\mathfrak{p}_S$ is a parabolic subalgebra with nilradical $\mathfrak{m}_S$. Let $\mathfrak{m} = {}^t\mathfrak{m}_S$; we have $\mathfrak{g} = \mathfrak{p}_S \oplus \mathfrak{m}$.

Let $P_S^+ = \{\lambda \in \mathfrak{h}^* | (\lambda, \check{\alpha}) \in \mathbf{N} \text{ for } \alpha \in S\}$, the set of weights which are dominant integral on restriction to $\mathfrak{h}_S$. The Weyl group associated to R will be denoted by W, with reflections s_α corresponding to roots α, and the dot action of W on $\mathfrak{h}^*$ is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half-sum of the positive roots. For λ in R, we define $R_\lambda = \{\alpha \in R | (\lambda, \check{\alpha}) \in \mathbf{Z}\}$ and W_λ is the subgroup of W generated by $\{s_\alpha | \alpha \in R_\alpha\}$. There is a unique base B_λ of R_λ lying in R^+. Given a subset S of B, we denote by W_S the subgroup generated by $\{s_\alpha | \alpha \in S\}$; it is the Weyl group of R_S, and lies in W_λ if $S \subset B_\lambda$. The weight λ is *regular* with respect to W (or one of its subgroups) if the only w with $w \cdot \lambda = \lambda$ is the identity element e.

Let λ be an element of P_S^+. The groups W_S and W_λ both have the Bruhat order $<$. With respect to the length function l, any element $w \in W_\lambda$ factors uniquely as $(w')({}^Sw)$ with $w' \in W_S$, $l(w) = l(w') + l({}^Sw)$, and ${}^Sw$ of minimal length in the coset W_Sw [**4**, p. 37]. The coset space ${}^SW_\lambda = W_S \backslash W_\lambda$ inherits a Bruhat order by identifying it with the set $\{{}^Sw | w \in W_\lambda\}$ and restricting to this set the Bruhat order on all of W_λ. Let w_0 denote the longest element of W_λ and ${}^Sw_0$ the corresponding maximal element of ${}^SW_\lambda$ under the Bruhat order. Let w_S denote the longest element of W_S.

2.1 Given a weight μ in $\mathfrak{h}^*$, we denote the Verma module of highest weight μ by $M(\mu)$ and its simple top by $L(\mu)$. Given a subset S of B and $\mu \in P_S^+$, we let $L^S(\mu)$ be the finite-dimensional $\mathfrak{p}_S$-module defined as a $\mathfrak{g}_S$-module to be $L(\mu|\mathfrak{h}_S)$, and extended to $\mathfrak{p}_S$ with $\mathfrak{m}_S$ acting trivially and $\mathfrak{h}$ acting via μ. The generalized

Verma module $M_S(\mu)$ is the module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} L^S(\mu)$. By a result of Lepowsky [**12**, 2.4], $M_S(\mu)$ is the homomorphic image of $M(\mu)$ modulo the sum of the Verma submodules $M(s_\alpha \cdot \mu)$ for $\alpha \in S$. Given weights μ and η with $M(\eta) \subset M(\mu)$, a homomorphism $\sigma\colon M_S(\eta) \to M_S(\mu)$ is induced, which may be zero. We call σ the standard map (it is only defined up to scalar multiple); if σ is nonzero, we call the composition factor corresponding to $\sigma(M_S(\eta))/\sigma(\operatorname{rad} M_S(\eta))$ a *standard* composition factor of $M_S(\mu)$. The standard composition factors can be explicitly determined in terms of the Bruhat order on W_μ, via Lepowsky's result.

The category $\mathcal{O}_S$ is the full subcategory of $U(\mathfrak{g})$-modules which are finitely-generated, $U(\mathfrak{m})$-finite, and become direct sums of finite-dimensional simple modules on restriction to $U(\mathfrak{g}_S + \mathfrak{h})$. All the modules have finite length, with composition factors forming the set $\{L(\mu) | \mu \in P_S^+\}$. The projective cover $P_S(\mu)$ of $L(\mu)$ has a Verma flag, and given η in P_S^+, the number of occurrences $[P_S(\mu) : M_S(\eta)]$ of $M_S(\eta)$ as a factor in a Verma flag is independent of the flag, with

$$[P_S(\mu) : M_S(\eta)] = (M_S(\eta) : L(\mu)).$$

These facts can be found in [**1**; **13**, 6.1]. If $S = \varnothing$, we omit the S from all notations.

A weight μ is dominant (antidominant) if μ is maximal (minimal) in the set $W_\mu \cdot \mu$ (under the usual ordering of $\mathfrak{h}^*$). Let λ be a dominant weight in P_S^+. By the reciprocity principle above, $M_S(\lambda)$ is projective. Let $\mathcal{O}_S^\lambda$ be the full subcategory of $\mathcal{O}_S$ consisting of modules all of whose composition factors have highest weight in $W_\lambda \cdot \lambda$. Every module of $\mathcal{O}_S$ is a direct sum of modules in $\mathcal{O}_S^\lambda$, as λ runs through dominant weights of P_S^+, and each $\mathcal{O}_S^\lambda$ is a block of $\mathcal{O}_S$. This follows for general S from the case $S = \varnothing$, due to Bernstein, Gelfand and Gelfand (BGG) [**1**]. Each block has only finitely many simple modules, with highest weights forming the set ${}^S W_\lambda \cdot \lambda$.

A duality D is defined on all of these categories. Given M in $\mathcal{O}$, the dual space M^* is a $\mathfrak{g}$-module via the transposition t, and DM is the submodule $\bigoplus_{\mu \in h^*} (M_\mu)^*$ of M^*. The module DM lies in $\mathcal{O}$, and because D peserves weight space dimensions, DM and M have the same composition factors. In particular, $DP_S(\mu)$ is the injective envelope of $L(\mu)$ in $\mathcal{O}_S$. Homomorphisms $\phi\colon M \to DM$ correspond bijectively to contravariant, symmetric bilinear forms on M, the form being nondegenerate if and only if ϕ is an isomorphism. Hence, M is self-dual if and only if M has a nondegenerate, symmetric, contravariant bilinear form.

2.2 The Jantzen translation functors T_μ^η are defined as usual [**9**, 2.10]. Given a dominant, regular weight λ and $\alpha \in B_\lambda$, the functor θ_α of translation across the α-wall is defined on $\mathcal{O}^\lambda$ as in [**6**]. We summarize the basic facts on θ_α, which are fundamental to all the results of this paper.

PROPOSITION. *Let w be an element of W_λ with $ws_\alpha < w$.*

(i) *$\theta_\alpha M(w\cdot\lambda)$ is an extension of $M(ws_\alpha\cdot\lambda)$ by $M(w\cdot\lambda)$ with simple top $L(w\cdot\lambda)$, and coincides with $\theta_\alpha M(ws_\alpha \cdot \lambda)$.*

(ii) *$\theta_\alpha L(w \cdot \lambda)$ is self-dual with top and socle equal to $L(w \cdot \lambda)$.*

(iii) *$\theta_\alpha L(w\cdot\lambda)$ has $L(w\cdot\lambda)$ as a composition factor with multiplicity* 2, *$L(ws_\alpha\cdot\lambda)$ with multiplicity* 1, *and $L(w \cdot \lambda)$ is the only composition factor not annihilated by θ_α.*

(iv) *The maximal proper submodule $U_\alpha L(w\cdot\lambda)$ of $\theta_\alpha L(w\cdot\lambda)/L(w\cdot\lambda)$ is semisimple.*

REMARKS. Proofs of (i)–(iii) can be found in [**6**, 3.6]. Part (iv) is Vogan's Conjecture [**16**, 2.5], and is equivalent to the Kazhdan-Lusztig Conjecture, which is a theorem. (The integral case was proved by Beilinson-Bernstein and Brylinski-Kashiwara. See [**14**].)

We will want to apply θ_α to $\mathcal{O}_S^\lambda$ as well, for $\lambda \in P_S^+$, and must make sure this makes sense. Given a finite-dimensional $\mathfrak{g}$-module E and M in $\mathcal{O}_S$, the module $M \otimes E$ is in $\mathcal{O}_S$ as well, from which it follows that θ_α sends $\mathcal{O}_S^\lambda$ to $\mathcal{O}_S^\lambda$. In particular, for $w \in {}^S W_\lambda$, the modules $\theta_\alpha L(w \cdot \lambda)$ and $U_\alpha L(w\cdot\lambda)$ lie in $\mathcal{O}_S^\lambda$. The usual argument (see [**9**, 2.2 and **5**, 5.3]) shows that if M is a generalized Verma module, then $M\otimes E$ has a (generalized) Verma flag, and given $w \in {}^S W_\lambda$ with $ws_\alpha < w$, the flag of $\theta_\alpha M_S(w\cdot\lambda)$ has two constituents, $M_S(w\cdot\lambda)$ and $M_S(ws_\alpha\cdot\lambda)$. Using the exactness of θ_α, we obtain the analogue of (iv):

PROPOSITION (v). *Given λ dominant regular in P_S^+ and $w \in {}^S W_\lambda$ with $ws_\alpha < w$, the module $\theta_\alpha M_S(w\cdot\lambda)$ is an extension of $M_S(ws_\alpha\cdot\lambda)$ by $M_S(w\cdot\lambda)$ with simple top $L(w\cdot\lambda)$.*

We also need the analogue for $\mathcal{O}_S$ of a result for $\mathcal{O}$ on the behavior of projectives under θ_α [**11**, 3.8]. Given P projective in $\mathcal{O}_S$ and E finite-dimensional, $P\otimes E$ is projective in $\mathcal{O}_S$, as the argument of [**9**, 2.24] proves.

PROPOSITION (vi). *Given λ dominant regular in P_S^+ and $w \in {}^S W_\lambda$ with $ws_\alpha > w$, the module $\theta_\alpha P_S(w\cdot\lambda)$ equals $P_S(ws_\alpha\cdot\lambda)\oplus P'$ for some projective module P' of $\mathcal{O}_S$. The indecomposable summands of P' are of the form $P_S(y\cdot\lambda)$ for $y < ws_\alpha$.*

PROOF. By the remark before Proposition (vi), $\theta_\alpha P_S(w\cdot\lambda)$ is projective, and it inherits from $P_S(w\cdot\lambda)$ a Verma flag with two constituents $M_S(y\cdot\lambda)$ and $M_S(ys_\alpha\cdot\lambda)$ for each $M_S(y\cdot\lambda)$ in $P_S(w\cdot\lambda)$. We are using here Proposition (v) and the exactness of θ_α. In addition, each constituent $M_S(y\cdot\lambda)$ of $P_S(w\cdot\lambda)$ satisfies $y \le w$, with exactly one constituent of the form $M_S(w\cdot\lambda)$, by the reciprocity principle of §2.1, and this constituent is at the top. Therefore, the flag of $\theta_\alpha P_S(w\cdot\lambda)$ has $M_S(ws_\alpha\cdot\lambda)$ exactly once, at its top, and the proposition follows. □

Finally, given M in $\mathcal{O}$ and E finite-dimensional, the isomorphism $DE\cong E$ yields $D(M\otimes E)\cong DM\otimes E$, and so if M is self-dual, $M\otimes E$ is as well. By [**9**, 2.3] and the equivalence between self-duality and the existence of a nondegenerate, symmetric, contravariant bilinear form, we deduce that T_μ^η and θ_α preserve the property of self-duality.

3. Self-duality in the category $\mathcal{O}$. The Conjecture of the introduction is easily proved for the category $\mathcal{O}$ by using the results recorded in §2.2. Given λ dominant in $\mathfrak{h}^*$ and $w \in W_\lambda$, the socle of $M(w\cdot\lambda)$ is $L(w_0\cdot\lambda)$, a result of Borho (see the proof in §4.1). Thus the Conjecture reduces to the following statement:

THEOREM. *Given an antidominant weight ν in $\mathfrak{h}^*$, the module $P(\nu)$ is isomorphic to $DP(\nu)$.*

PROOF. (i) We may assume ν is regular. In case it is not, there is an antidominant, regular weight η with $\eta-\nu$ dominant integral, and we can apply T_η^ν to $P(\eta)$. Assuming we know $P(\eta)$ is self-dual, so is $T_\eta^\nu P(\eta)$ by §2.2, and it is a nonzero projective. Therefore $T_\eta^\nu P(\eta)$ is a direct sum of modules $P(y\cdot\nu)$ with $y\in W_\nu$. But the socle is a direct sum of copies of $L(\nu)$, since each $P(y\cdot\nu)$ has a Verma flag, and its constituent Verma modules have socle $L(\nu)$. The top and socle of $T_\eta^\nu P(\eta)$ are isomorphic, so the top too is a direct sum of $L(\nu)$'s, and each summand $P(y \cdot \nu)$ must be $P(\nu)$ itself. Hence $P(\nu)$ is self-dual.

(ii) Assume ν is regular for the remainder of the proof, and let $\lambda = w_0 \cdot \nu$, the dominant weight in $W_\nu \cdot \nu$. Choose a minimal length factorization $s_{\alpha_1} \cdots s_{\alpha_r}$ of w_0 with $\alpha_i \in B_\nu$ and let θ_0 be the functor $\theta_{\alpha_1} \cdots \theta_{\alpha_r}$. The antidominance of ν means that $s_\alpha \cdot \nu > \nu$ for all $\alpha \in B_\nu$. Therefore, for $w \in W_\nu$, the module $\theta_\alpha L(y \cdot \nu)$ does not have $L(\nu)$ as a composition factor unless $y = e$, in which case $L(\nu)$ occurs twice (§2.2(iii)). From this and §2.2(i), or [**9**, 2.16], one obtains the well-known fact that $(M(\lambda) : L(\nu)) = 1$. In addition, since the one appearance of $L(\nu)$ in $M(\lambda)$ is as a submodule, $\theta_0 L(\nu)$ is a submodule of $\theta_0 M(\lambda)$. An $L(\nu)$ can arise as a composition factor in either module only by successive applications of the θ_{α_i}'s to appearances of $L(\nu)$, so that we obtain by induction

$$(\theta_0 M(\lambda) : L(\nu)) = (\theta_0 L(\nu) : L(\nu)).$$

(iii) As noted in §2.1, the dominance of λ implies that $M(\lambda) = P(\lambda)$. Iterated use of §2.2(vi) yields $\theta_0 M(\lambda) = P(\nu) \oplus K$ for a projective module K in which $P(\nu)$ is not a summand. The top of $P(\nu)$ lies in the submodule $\theta_0 L(\nu)$ by (ii), so that $P(\nu)$ is itself a submodule of $\theta_0 L(\nu)$. Hence

$$\theta_0 L(\nu) = P(\nu) \oplus (K \cap \theta_0 L(\nu)).$$

By the remark at the end of §2.2, the module $\theta_0 L(\nu)$ is self-dual, so $DP(\nu)$ is also a summand. But $DP(\nu)$ is an injective module and as a submodule of $\theta_0 L(\nu)$, it is also a submodule of $\theta_0 M(\lambda)$, so $DP(\nu)$ is a summand of $\theta_0 M(\lambda)$. Because $\theta_0 M(\lambda)$ is projective, $DP(\nu)$ is an indecomposable projective module, implying that the top is simple. But the top coincides with the socle of $P(\nu)$, which must be a direct sum of copies of $L(\nu)$. We may conclude that $DP(\nu)$ has $L(\nu)$ as its top, and is isomorphic to $P(\nu)$. □

4. Self-duality in $\mathcal{O}_S$. In this section we will see how the Conjecture for $\mathcal{O}_S^\lambda$ follows from an assumption about the action of the functors θ_α on $M_S(\lambda)$. We also reformulate the assumption in a stronger form which lends itself more easily to verification and which, for $\mathcal{O}^\lambda$, amounts to the statement $(M(\lambda) : L(w_0 \cdot \lambda)) = 1$. In effect, the assumption plays the role of step (ii) in the proof of §3. We fix a subset S of B in this section.

4.1 Fundamental to all we do is the following fact:

PROPOSITION. *Let $\lambda \in P_S^+$ be dominant regular. Then $M_S(\lambda)$ has simple socle.*

PROOF. (i) Assume $\lambda = 0$ on $\mathfrak{h}_S$. Then $M_S(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} L^S(\lambda)$ with $L^S(\lambda)$ one-dimensional. Hence $M_S(\lambda)$ is isomorphic to $U(\mathfrak{m})$ as a $U(\mathfrak{m})$-module. Since $U(\mathfrak{m})$ is an Ore domain, any two nonzero submodules of $M_S(\lambda)$ have nonzero intersection, and the socle is simple.

(ii) Let η be the unique weight with $(\eta, \check{\alpha}) = 0$ for $\alpha \notin S$ and $(\eta, \check{\alpha}) = (\lambda, \check{\alpha})$ for $\alpha \in S$, and let $\lambda' = \lambda - \eta$. Then $\lambda' = 0$ on $\mathfrak{h}_S$ and $M_S(\lambda')$ has simple socle by (i). But λ and λ' are in the same facette, so $T_{\lambda'}^\lambda$ is a category equivalence from $\mathcal{O}^{\lambda'}$ to $\mathcal{O}^\lambda$ [**9**, 2.15], with $T_{\lambda'}^\lambda M_S(\lambda') = M_S(\lambda)$. Therefore $M_S(\lambda)$ has simple socle as well. □

REMARK. The argument of (i) is Borho's argument for the simplicity of the socle of any Verma module. As remarked in the introduction, generalized Verma modules need not have simple socle.

4.2 We will also need the following elementary result:

PROPOSITION. *Given a projective module P in $\mathcal{O}_S$ which is isomorphic to DP, each indecomposable summand of P is also self-dual.*

PROOF. Let $P = Q \oplus K$ with Q indecomposable. The dual module DQ must also be a summand of P, and so is a projective indecomposable. The modules Q and DQ have the same composition factors and by the reciprocity principle (§2.1) a projective indecomposable is determined up to isomorphism by its composition factors. Hence $Q \cong DQ$. □

4.3 Let λ be a dominant, regular weight in P_S^+, and let ${}^S X_\lambda = \{y \in {}^S W_\lambda | L(y \cdot \lambda)$ is a summand of the socle of $M_S(w \cdot \lambda)$ for some $w \in {}^S W_\lambda\}$. In the terminology of the introduction, ${}^S X_\lambda$ is the set of socular elements of ${}^S W_\lambda$. This set admits a precise characterization, pointed out by David Vogan, the proof of which depends on several results of A. Joseph (and O. Gabber). For convenience, we will refer to Jantzen's book for all of these results [**10**], and adopt his notation. Given simple modules $L(\mu)$ and $L(\eta)$, recall that $\mathrm{Hom}_C(L(\mu), L(\eta))$ is a $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule, which on restriction becomes a module over the diagonal subalgebra $\mathfrak{k} = \{(x, -{}^t x) | x \in \mathfrak{g}\}$ of $\mathfrak{g} \times \mathfrak{g}$, and that $\mathcal{L}(L(\mu), L(\eta))$ is the submodule of $\mathfrak{k}$-finite vectors. Also, $I(\mu) = \mathrm{ann}_{U(\mathfrak{g})} L(\mu)$ and $d(M)$ is the Gelfand-Kirillov dimension of M.

PROPOSITION. *The following conditions on an element y of W_λ are equivalent.*

(i) *$L(y \cdot \lambda)$ is a summand of the socle of a generalized Verma module in $\mathcal{O}_S^\lambda$.*

(ii) *There is a finite-dimensional $\mathfrak{g}$-module E such that*

$$\mathrm{Hom}_{\mathfrak{g}}(L(y \cdot \lambda), L({}^S w_0 \cdot \lambda) \otimes E)$$

is nonzero.

(iii) $\mathcal{L}(L(y \cdot \lambda), L({}^S w_0 \cdot \lambda)) \neq 0$.

(iv) $I(y^{-1} \cdot \lambda) = I({}^S w_0^{-1} \cdot \lambda)$.

(v) *y is in ${}^S W_\lambda$ and* $d(L(y \cdot \lambda)) = d(L({}^S w_0 \cdot \lambda))$.

PROOF. (a) The equivalence of (i) and (ii) is elementary. Given a finite-dimensional module E, since $L({}^S w_0 \cdot \lambda) = M_S({}^S w_0 \cdot \lambda)$, the module $L({}^S w_0 \cdot \lambda) \otimes E$ has a (generalized) Verma flag in $\mathcal{O}_S$ (§2.2). Thus, under the assumption of (ii), the simple $L(y \cdot \lambda)$ is in the socle of a bottom constituent in this flag. Conversely, any $M_S(w \cdot \lambda)$ is a submodule of $M_S({}^S w_0 \cdot \lambda) \otimes E$ for some finite-dimensional module E, as follows from §2.2, Proposition (v) by applying a suitable sequence of θ_α's to $M_S({}^S w_0 \cdot \lambda)$.

(b) By [**10**, 6.8], we have

$$\mathrm{Hom}_{\mathfrak{g}}(L(y \cdot \lambda), L({}^S w_0 \cdot \lambda) \otimes E) \cong \mathrm{Hom}_{\mathfrak{k}}(E^*, \mathcal{L}(L(y \cdot \lambda), L({}^S w_0 \cdot \lambda))).$$

Thus $\mathcal{L}(L(y \cdot \lambda), L({}^S w_0 \cdot \lambda))$ is nonzero if and only if (ii) holds.

(c) The equivalence of (iii) and (iv) is a theorem of Gabber and Joseph [**10**, 7.14].

(d) For a simple module $L(w \cdot \lambda)$, we have $2d(L(w \cdot \lambda)) = d(U(\mathfrak{g})/I(w \cdot \lambda)) = d(U(\mathfrak{g})/I(w^{-1} \cdot \lambda))$, by [**10**, 10.8 and 10.10b], from which it follows that (iv) implies (v). For the converse, by [**10**, 5.18], we find for $y \in {}^S W_\lambda$ that $I({}^S w_0 \cdot \lambda) \subset I(y \cdot \lambda)$, yielding the implication. $\square$

REMARK. Another way to express the equivalence of (i) and (ii) is that ${}^S X_\lambda$ is precisely the subset $\{w \in W_\lambda | w \sim_R {}^S w_0\}$; that is ${}^S X_\lambda$ is the right cell of W_λ containing ${}^S w_0$. The Kazhdan-Lusztig conjectures provide an algorithm for computing right cells [**10**, 16.4], hence for computing ${}^S X_\lambda$, and we will use this in §9.2.

4.4 As noted in the introduction, we will not use these results in our verification of the Conjecture until §9. For our approach until then, we need another order on ${}^S W_\lambda$, weaker than the Bruhat order: $y \ll w$ if and only if there are simple roots $\beta_1, \ldots, \beta_t$ in B_λ with $y < ys_{\beta_1} < \cdots < ys_{\beta_1} \cdots s_{\beta_t} = w$. Let ${}^S X'_\lambda$ be the set of elements in ${}^S X_\lambda$ which are minimal under the $\ll$ order. We will see in §6.4 that the minimal elements of ${}^S X_\lambda$ with respect to the $<$ and $\ll$ orders may differ. Let ${}^S Y_\lambda = \{y \in {}^S W_\lambda | P_S(y \cdot \lambda) \text{ is self-dual}\}$; we have ${}^S Y_\lambda \subset {}^S X_\lambda$, and the Conjecture states that they are equal. Let $L(\nu)$ be the socle of $M_S(\lambda)$. For each $w \in {}^S W_\lambda$, choose a factorization $s_{\alpha_1} \cdots s_{\alpha_r}$ of minimal length and define θ_w to be the functor $\theta_{\alpha_r} \circ \cdots \circ \theta_{\alpha_1}$.

DEFINITION. An element w of ${}^S W_\lambda$ is *special* if there is a minimal factorization of w for which θ_w satisfies

$$(\theta_w M_S(\lambda) : L(w \cdot \lambda)) = (\theta_w L(\nu) : L(w \cdot \lambda)).$$

In other words, w is special if all the occurrences of $L(w \cdot \lambda)$ in $\theta_w M_S(\lambda)$ lie in the submodule $\theta_w L(\nu)$. By §2.2(vi), the module $P_S(w \cdot \lambda)$ is a summand of $\theta_w M_S(\lambda)$, so that the multiplicity in question is not 0.

THEOREM. *Let λ be a dominant, regular weight in P_S^+. If $w \in {}^S W_\lambda$ is special, then $P_S(y \cdot \lambda)$ is self-dual for all $y \geq\geq w$.*

PROOF. (i) By §2.2(vi), the module $\theta_w M_S(\lambda) = P_S(w \cdot \lambda) \oplus K$ for a projective K with summands $P_S(y \cdot \lambda)$ satisfying $y < w$. By specialness, $P_S(w \cdot \lambda)$ is a submodule of $\theta_w L(\nu)$, and

$$\theta_w L(\nu) \cong P_S(w \cdot \lambda) \oplus (K \cap \theta_w L(\nu)).$$

Since $\theta_w L(\nu)$ is self-dual, the injective module $DP_S(w \cdot \lambda)$ is a submodule as well, and must be isomorphic to a summand of $\theta_w M_S(\lambda)$. Therefore $DP_S(w \cdot \lambda)$ is a projective indecomposable with the same composition factors as $P_S(w \cdot \lambda)$, and the two are isomorphic.

(ii) Given $\alpha \in B_\lambda$ with $ws_\alpha > w$, we obtain a self-dual projective module $\theta_\alpha P_S(w \cdot \lambda)$ with $P_S(ws_\alpha \cdot \lambda)$ as a summand, by §2.2(vi). By §4.2, the module $P_S(ws_\alpha \cdot \lambda)$ is self-dual. Continuing in this fashion, we obtain the self-duality of $P_S(y \cdot \lambda)$ for all $y \gg w$. $\square$

COROLLARY. *Let λ be a dominant, regular weight in P_S^+. If every element of ${}^S X'_\lambda$ is special, then $P_S(y \cdot \lambda)$ is self-dual for all $y \in {}^S X_\lambda$.* $\square$

4.5 In order to use the Theorem, we need to be able to find special elements. We describe next a condition implying specialness which is easier to check. Given $w \in W_\lambda$, let $\tau_\lambda(w) = \{\alpha \in B_\lambda | ws_\alpha < w\}$. Let $\overline{w}$ denote $w_{\tau_\lambda(w)}$, the longest element in the Weyl group $W_{\tau_\lambda(w)}$.

LEMMA. *An element $w \in W_\lambda$ factors as $(w\overline{w})\overline{w}$, with $l(w\overline{w}) + l(\overline{w}) = l(w)$.*

PROOF. Let $T = \tau_\lambda(w)$ and consider the right coset space $W^T = W_\lambda/W_T$. The analogue of a result in §2.0 yields a unique factorization $w = (w^T)(w')$ with w' in W_T and w^T of minimal length among elements of wW_T, such that $l(w) = l(w^T) + l(w')$. For each $\alpha \in T$, we have $l(ws_\alpha) < l(w)$ and $ws_\alpha W_T = wW_T$. Thus we must have $l(w's_\alpha) < l(w')$ for all $\alpha \in T$, and w' is the longest element $\overline{w}$ of W_T. The result follows. $\square$

PROPOSITION. *Let w be an element of ${}^SW_\lambda$ such that for some minimal factorization of $w\overline{w}$, the functor $\theta_{w\overline{w}}$ satisfies*

$$(\theta_{w\overline{w}} M_S(\lambda) \colon L(w \cdot \lambda)) = (\theta_{w\overline{w}} L(\nu) : L(w \cdot \lambda)).$$

Then w is special.

PROOF. By the Lemma, we obtain a minimal factorization of w by juxtaposition of minimal factorizations for $w\overline{w}$ and $\overline{w}$, yielding $\theta_w = \theta_{\overline{w}} \circ \theta_{w\overline{w}}$. By assumption, $L(w \cdot \lambda)$ has multiplicity 0 in $\theta_{w\overline{w}}(M_S(\lambda)/L(\nu))$. Each factor θ_α of $\theta_{\overline{w}}$ satisfies $ws_\alpha < w$, so that $L(w \cdot \lambda)$ occurs as a composition factor of $\theta_\alpha L(y \cdot \lambda)$ only for $y = w$. Therefore $L(w \cdot \lambda)$ has multiplicity 0 also in $\theta_{\overline{w}}\theta_{w\overline{w}}(M_S(\lambda)/L(\nu))$. $\square$

4.6 There is a connection between self-dual projectives and certain generalized Verma modules with simple socle, which we discuss in this subsection.

PROPOSITION 1. *Let λ be a dominant weight in P_S^+ and w an element of ${}^SW_\lambda$ for which $P_S(w \cdot \lambda)$ is self-dual. The set $\{y \in {}^SW_\lambda | (M_S(y \cdot \lambda) : L(w \cdot \lambda)) > 0\}$ contains a unique minimal element $\check{w}$ with respect to the Bruhat order. Moreover, $M_S(\check{w} \cdot \lambda)$ has simple socle $L(w \cdot \lambda)$ and $(M_S(\check{w} \cdot \lambda) : L(w \cdot \lambda)) = 1$.*

PROOF. Given elements y, z in ${}^SW_\lambda$ with $y \not< z$, there can be no non-split extension of $M_S(y \cdot \lambda)$ by $M_S(z \cdot \lambda)$, since any vector of weight $z \cdot \lambda$ in an extension of the two modules must be a highest weight vector. Therefore, for each minimal element z in the set of the proposition, there is a Verma flag of $P_S(w \cdot \lambda)$ with $M_S(z \cdot \lambda)$ as the lowest submodule. Hence, given a list $\{z_1, \dots, z_r\}$ of minimal elements, with each z repeated $(M_S(w \cdot \lambda) : L(z \cdot \lambda))$ times, $P_S(\lambda)$ has $\bigoplus_{i=1}^r M_S(z_i \cdot \lambda)$ as a submodule. Since $P_S(w \cdot \lambda)$ has $L(w \cdot \lambda)$ as socle, the conclusion follows. $\square$

Let ${}^SZ_\lambda = \{\check{w} \in {}^SW_\lambda | P_S(w \cdot \lambda) \text{ is self-dual}\}$. This set is restricted in the following way.

PROPOSITION 2. *Given $\check{w} \in {}^SZ_\lambda$ and $\alpha \in B_\lambda$ with $\check{w}s_\alpha < w$, the element $\check{w}s_\alpha$ lies in ${}^SZ_\lambda$.*

PROOF. The proof of Proposition 1 shows that $P_S(w \cdot \lambda)$ has $M_S(\check{w} \cdot \lambda)$ at the bottom of any Verma flag. By §2.2(v), a Verma flag for $\theta_\alpha P_S(w \cdot \lambda)$ has $M_S(\check{w}s_\alpha \cdot \lambda)$ at the bottom. In particular, some summand $P_S(y \cdot \lambda)$ of $\theta_\alpha P_S(w \cdot \lambda)$ has a Verma flag with $M_S(\check{w}s_\alpha \cdot \lambda)$ at the bottom. By §4.2, the module $P_S(y \cdot \lambda)$ is self-dual, and $\check{w}s_\alpha = \check{y}$. $\square$

REMARK. The proposition states that ${}^SZ_\lambda$ is an initial subset of ${}^SW_\lambda$ under $\ll$, while by §4.2 the set ${}^SY_\lambda$ is a final subset of ${}^SW_\lambda$, and one may imagine a closer correspondence between the two sets. We can describe one possibility explicitly.

The Bruhat poset ${}^SW_\lambda$ has an anti-automorphism given by the map $y \mapsto y^* = w_0^{-1}({}^Sw_0)^{-1}yw_0$ [**15**, p. 181]. This map also yields an anti-automorphism of ${}^SW_\lambda$ as poset with respect to $\ll$. To see this, suppose $y < ys_\alpha$ with $\alpha \in B_\lambda$. Then $y^* > (ys_\alpha)^*$ and we are claiming that $y^* \gg (ys_\alpha)^*$. Computing yields $(ys_\alpha)^* = w_0^{-1}({}^Sw_0)^{-1}yw_0(w_0^{-1}s_\alpha w_0) = y^* s_{w_0(\alpha)}$, so that $y^* > y^* s_{w_0(\alpha)}$. Since $w_0(B_\lambda) = -B_\lambda$, the claim follows.

A natural guess is that ${}^SY_\lambda$ and ${}^SZ_\lambda$ correspond under the anti-automorphism $*$. A more precise formulation is the following.

QUESTION. *Let λ be dominant regular in P_S^+ and let w be an element of ${}^SY_\lambda$. Is it true that $(\breve{w})^* = w^*$? In other words, if $P_S(w \cdot \lambda)$ is self-dual with $\breve{w} \cdot \lambda$ as the highest weight of any of its vectors, is $P_S((\breve{w})^* \cdot \lambda)$ self-dual with highest weight $w^* \cdot \lambda$?*

In all the examples in which we can answer this question, the answer is yes.

4.7 We now describe the procedure mentioned in the introduction for verifying the Conjecture, in terms of the notation since introduced. We fix a dominant regular weight λ in P_S^*.

(1) Determine the composition factors and their multiplicities in $M_S(w \cdot \lambda)$ for all $w \in {}^SW_\lambda$.

(2) Determine the socle $L(\nu)$ of $M_S(\lambda)$.

(3) Using (1), (2) and §4.5, verify that some subset X' of ${}^SW_\lambda$ is special.

(4) By §4.4, obtain the set $X = \{y \in {}^SW_\lambda | y \gg x \text{ for some } x \in X'\}$ as a subset of ${}^SY_\lambda$, and by §4.6 obtain the subset $Z = \{\breve{w} | w \in X\}$ of ${}^SZ_\lambda$.

(5) Show for $w \notin Z$ that the socle of $M_S(w \cdot \lambda)$ is a direct sum of simples $L(y \cdot \lambda)$ with $y \in X$.

(6) Conclude that $X = {}^SX_\lambda = {}^SY_\lambda$ and the Conjecture is correct.

4.8 The remainder of the paper is devoted to a series of examples for which the Conjecture is verified. For simplicity, we will consider *only* integral weights λ. Thus W_λ will equal W. If λ and λ' are both regular integral weights, then $\mathcal{O}^\lambda$ and $\mathcal{O}^{\lambda'}$ are equivalent categories [**1**, Theorem 4], as are $\mathcal{O}_S^\lambda$ and $\mathcal{O}_S^{\lambda'}$. Hence ${}^SX_\lambda = {}^SX_{\lambda'}$, and similarly for the other subsets of ${}^SW$, so we may omit the subscript λ from these subsets. Moreover, no harm is done in writing $L(w), M_S(w), P_S(w)$ for the corresponding objects associated to the weight $w\cdot\lambda$. If no λ is mentioned, it is always tacitly assumed in any fixed example that λ is chosen to be dominant regular in P_S^+.

In the diagrams of Bruhat posets that appear at the end, the element e is always at the top, and an edge connecting w and ws_α with $w < ws_\alpha$ and $\alpha \in B$ will be marked by an α. The $\ll$ order can be obtained by omitting the unmarked edges. Actually, an element w of ${}^SW$ will typically be indicated in the diagrams by a number or other symbol, and we will freely write $M_S(\spadesuit)$, for example, if $\spadesuit$ corresponds to w.

5. A_n examples. Throughout this section, R is of type A_n with base $B = \{\alpha_1, \ldots, \alpha_n\}$ numbered in the standard way. We will write s_i and θ_i in place of s_{α_i} and θ_{α_i}.

5.1 Assume $S = \{\alpha_2, \ldots, \alpha_n\}$. The Bruhat order ${}^SW$ is in Diagram 1.

The structure of the generalized Verma modules is easily determined and well known [**3**, 4.6]: for $r \leq n$, the module $M_S(r)$ has length two with socle $L(r+1)$,

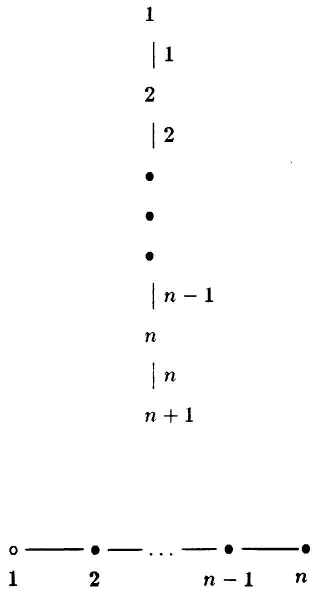


Diagram 1

and $M_S(n+1) = L(n+1)$. One can calculate this directly using [**9**, Chapter 5]. The projectives are easily described from this information, with $P_S(1) = M_S(1)$.

PROPOSITION. *For $2 \leq r \leq n+1$, the projective $P_S(r)$ equals $\theta_{r-1}L(r)$ and is self-dual.*

PROOF. We have $\theta_{r-1}L(r) = \theta_{r-1}M_S(r)$, and this is an extension of $M_S(r-1)$ by $M_S(r)$. Therefore $\theta_{r-1}L(r)$ has simple top and the same composition factors as $P_S(r)$, implying that they are isomorphic. $\square$

5.2 Assume $S = \{\alpha_2, \ldots, \alpha_{n-1}\}$. In this case, the composition factors of generalized Verma modules have been calculated by Boe and Collingwood [**2**], and the program of §4.7 succeeds. The Bruhat poset ${}^SW$ is depicted in Diagram 2, and the composition factors of generalized Verma modules are listed in Diagram 3, with the understanding that an entry (r,s) or $(r,s)^+$ should be omitted if $r = 0$ or $s = n+1$.

PROPOSITION. (i) *The socle of $M_S(0,1)$ is $L(0,1)^+$.*
(ii) ${}^SX' = \{(r, n-1-r)^+ | 0 \leq r \leq n-2\}$ *and all these elements are special.*
(iii) $P_S(\mu)$ *is self-dual for all $\mu \in {}^SX$.*

PROOF. (i) The four other composition factors of $M_S(0,1)$ are all standard, in the terminology of §2.1, so that their relative location in the socle series of $M_S(0,1)$ is clear, and the only other alternative for the socle is $L(1,2)$. Assume $L(1,2)$ actually is the socle.

The module $\theta_1 M_S(0,1)$ is an extension of $M_S(0,1)$ by $M_S(0,2)$ with $\theta_1 L(1,2)$ as a submodule. The top and bottom of $\theta_1 L(1,2)$ are $L(1,2)$, and the middle layer is $L(1,1) \oplus L(1,3)$ (this falls out in the calculation of generalized Verma modules, as described in [**2**]). The lower $L(1,2)$ corresponds to the socle of $M_S(0,1)$, and $L(1,3)$ is in the socle of $\theta_1 M_S(0,1)/L(1,2)$. Since $L(1,3)$ is not a composition factor of $M_S(0,1)$, it must be in the socle of $M_S(0,2)$.

The same argument can be repeated inductively on θ_r applied to $M_S(0,r)$ for $r \le n-2$, until we find that $M_S(0,n-1)$ has $L(0,n)^+ \oplus L(1,n-1)^+$ in its socle. But the only appearances of these simples in $M_S(0,n-1)$ are standard, and they lie above the standard appearance of $L(0,n-1)^+$ in the socle series, a contradiction.

(ii) The hypothesis of the Proposition in §4.5 is easily verified for each $(r,n-1-r)^+$. For example, as in the argument of (i), $\theta_{n-2}\cdots\theta_1 M_S(0,1)$ is an extension of $M_S(0,n-2)$ by $M_S(0,n-1)$. Also, since $L(0,1)^+ = M_S(0,1)^+$,

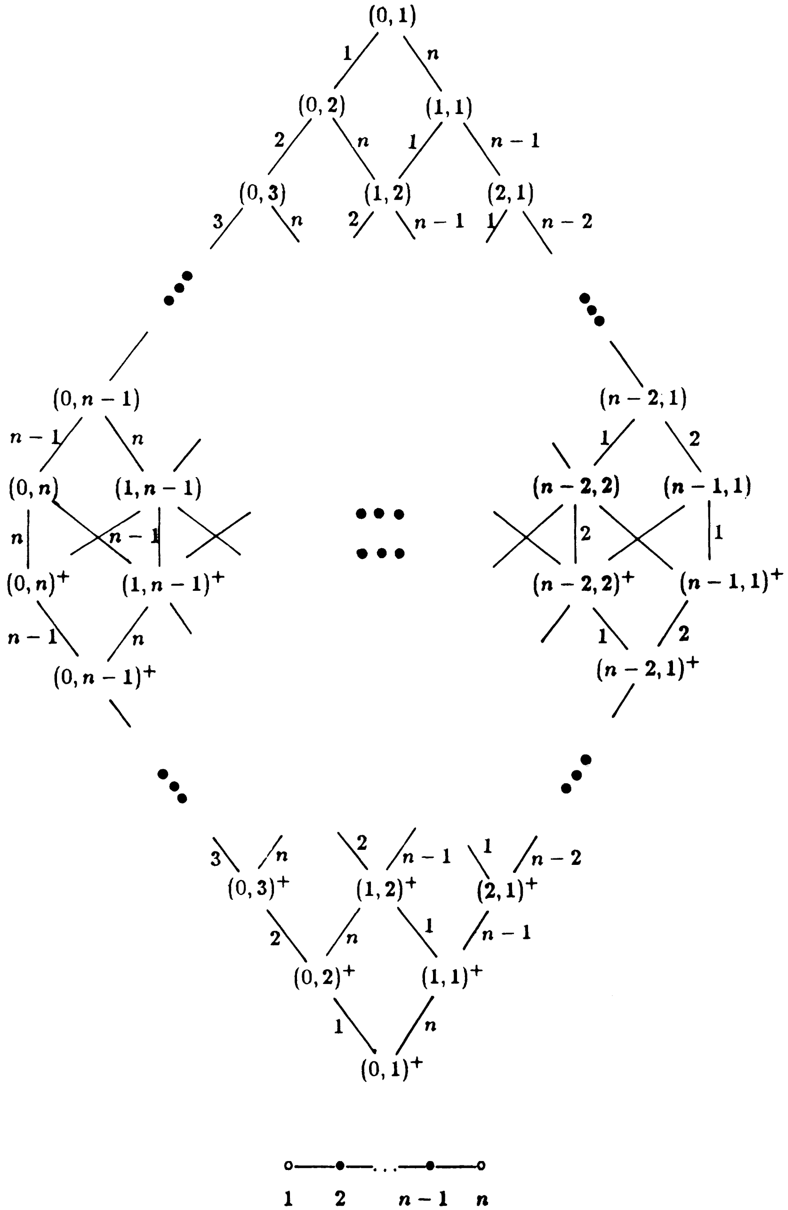


Diagram 2

$$\begin{array}{cccc} (r,s) & & & \\ (r,s+1)\quad (r+1,s) & (r-1,s-1)^+ & & \\ (r+1,s+1) & (r-1,s)^+ \quad (r,s-1)^+ & & (\mathbf{r+s<n-1}) \\ & (r,s)^+ & & \end{array}$$

$$\begin{array}{c} (r,n-1-r) \\ (r,n-r) \quad (r+1,n-1-r) \quad (r-1,n-2-r)^+ \\ (r,n-r)^+ \quad (r+1,n-1-r)^+ \quad (r-1,n-1-r)^+ \quad (r,n-2-r)^+ \\ (r,n-1-r)^+ \end{array}$$

$$\begin{array}{c} (r,n-r) \\ (r-1,n+1-r)^+ \quad (r,n-r)^+ \quad (r+1,n-1-r)^+ \quad (r-1,n-1-r)^+ \\ (r-1,n-r)^+ \quad (r,n-r-1)^+ \end{array}$$

$$\begin{array}{c} (r,s)^+ \\ (r-1,s)^+ \quad (r,s-1)^+ \\ (r-1,s-1)^+ \end{array}$$

Diagram 3

we find $\theta_{n-2}\cdots\theta_1 L(0,1)^+$ is an extension of $M_S(0,n-1)^+$ by $M_S(0,n-2)^+$. Both extensions involve $L(0,n-1)^+$ exactly once, which is the hypothesis of the Proposition in §4.5. Applying §§4.4 and 4.6, we find that $P_S(r,s)^+$ is self-dual for $r+s\leq n-1$ and $M_S(r,s)$ has simple socle $L(r,s)^+$.

This brings us to step (5) of §4.7. We must show that no other simples are socular, and the only generalized Verma modules which need to be examined are $M_S(w)$ for $w=(r,n-r)$ or $w=(r,n-r)^+$. These present no problem. All the composition factors of $M_S(r,n-r)^+$ are standard, and the socle is $L(r-1,n-r-1)^+$, unless $r=0,n$. The only nonstandard composition factor of $M_S(r,n-r)$ is $L(r-1,n-1-r)^+$, and an examination of the remaining standard composition factors shows that the socle is a submodule of $L(r-1,n-r)^+\oplus L(r,n-r-1)^+\oplus L(r-1,n-r-1)^+$, all the summands of which are in our claimed ${}^S X$. □

REMARK. Regarding the situation at the end of the proof, Boe and Collingwood show that the socle of $M_S(r,n-r)$ is the direct sum of the two standard composition factors. From our point of view, this could be proved as follows. If $L(r-1,n-r-1)^+$ is in the socle, then $\theta_{n-r-1}L(r-1,n-r-1)^+$ is a submodule of $\theta_{n-r-1}M_S(r,n-r)$, which is an extension of $M_S(r,n-r-1)$ by $M_S(r,n-r)$. Therefore $L(r-1,n-r-1)^+$ is in the socle of $M_S(r,n-r-1)$. But we saw in the proof (via §4.6) that the socle is $L(r, n-r-1)^+$. (One could also verify this by direct calculation with a conveniently chosen λ, computing the action of $\mathfrak{n}^+$ on the $(r, n-r-1)^+ \cdot \lambda$ weight space of $M_S(r, n-r)$.)

5.3 Assume $S = \{\alpha_3, \ldots, \alpha_n\}$. In this case the Conjecture will be verified without obtaining the composition factors of generalized Verma modules. Let $z = s_1 s_2 s_1$ in W.

PROPOSITION. (i) *The socle of* $M_S(\lambda)$ *is* $L(z \cdot \lambda)$.
(ii) z *is special and is the unique element of* ${}^S X'$.
(iii) $P_S(w \cdot \lambda)$ *is self-dual for all* $w \in {}^S X$.

PROOF. (i) In case $n = 2$, we are in the setting of the category $\mathcal{O}$, with $z = w_0$, and the results follow from §3. Assume $n > 2$ and the Proposition is valid for A_{n-1}. The Bruhat poset ${}^S W$ for A_n is obtained from the one for A_{n-1} by the addition of $2n$ points, as indicated in Diagram 4. All but one new w satisfies $ws_n < w$. Therefore for these w the multiplicity of $L(w \cdot \lambda)$ in the Verma modules $M(\lambda)$ and $M(s_n \cdot \lambda)$ is the same, and $(M_S(\lambda) : L(w \cdot \lambda)) = 0$ because $M_S(\lambda)$ is a homomorphic image of $M(\lambda)/M(s_n \cdot \lambda)$. The remaining w satisfies $ws_{n-1} < w$, so if $n > 3$ we still obtain $(M_S(\lambda) : L(w \cdot \lambda)) = 0$. In case $n = 3$, the problematic w is $s_2 s_1 s_3 s_2$ and $(M_S(\lambda) : L(s_2 s_1 s_3 s_2 \cdot \lambda)) = 1$. The socle of $M_S(\lambda)$ cannot be $L(s_2 s_1 s_3 s_2 \cdot \lambda)$ for if it is, applying θ_2 yields $L(s_2 s_1 s_3 \cdot \lambda)$ in the socle of $M_S(s_2 \cdot \lambda)$, which an examination of standard composition factors shows to be false. (Composition factor information for A_3 is available, for instance, from [**9**, 5.24].) We may conclude for $n \geq 3$ that $M_S(\lambda)$ has seven composition factors, with one nonstandard, and $L(z \cdot \lambda)$ is the socle.

(ii) The Proposition in §4.5 shows z is special, and we must show that any $w \in {}^S W$ which is not $\geq\geq z$ is not socular. In the passage from A_{n-1} to A_n, three new elements of this description are added, the ones circled in Diagram 4: $s_2 s_3 \cdots s_n, s_1 s_2 \cdots s_n, s_2 s_1 s_3 \cdots s_n$. A case-by-case check shows that none of these is socular. For instance, $L(s_2 \cdots s_n \cdot \lambda)$ has the same multiplicity in $M(w \cdot \lambda)$ and $M(ws_n \cdot \lambda)$ for any $w < s_2 \cdots s_n$. If $s_n w = ws_n$, it follows that $(M_S(w \cdot \lambda) : L(s_2 \cdots s_n \cdot \lambda)) = 0$. The only $w \in {}^S W$ still to consider are $s_2 \cdots s_n$ and $s_2 \cdots s_{n-1}$. For these w, an examination of the standard composition factors of $M_S(w \cdot \lambda)$ shows that $L(s_2 \cdots s_n \cdot \lambda)$ appears only as a standard factor and is not in the socle. A similar argument works in the other two cases. The same argument works for the threesomes of w's added in the passage from A_{m-1} to A_m, for $3 \leq m \leq n-1$, showing that they are not socular. It is obvious from a consideration of standard composition factors that the remaining five elements, all $< z$, are not socular. This proves (ii), from which (iii) follows. $\square$

5.4 The results of §§5.1–5.3 imply the Conjecture for all parabolics in A_3. We list additional A_n cases for which the Conjecture is true.

(i) A_4, with $S = \{\alpha_1, \alpha_3, \alpha_4\}$; in this case, $\nu = s_2 s_1 s_3 s_2 \cdot \lambda$ and ${}^S X' = \{s_2 s_1 s_3\}$.
(ii) A_4, with $S = \{\alpha_1, \alpha_4\}$; in this case $\nu = {}^S w_0 \cdot \lambda$ and

$${}^S X' = \{{}^S w_0 s_2 s_3,\ {}^S w_0 s_3 s_2\}.$$

(iii) A_5, with $S = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$; in this case $\nu = {}^S w_0 \cdot \lambda$ and

$${}^S X' = \{{}^S w_0 s_3 s_2 s_4\}.$$

In all the examples of this section, $\nu = w \cdot \lambda$ for w equal to ${}^S w_0$ or the unique element of ${}^S X'$. This is often not the case, as later examples show.

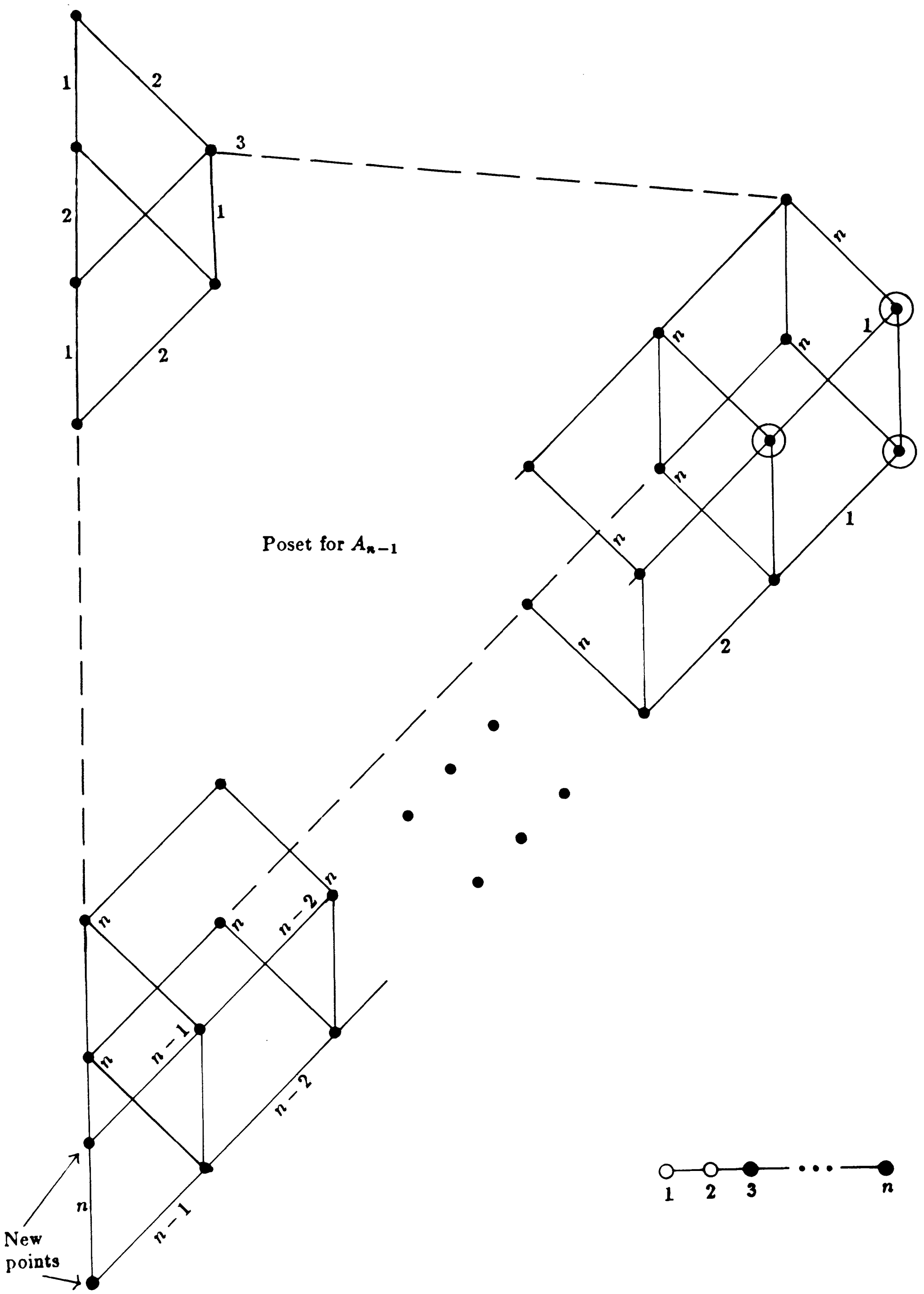


Diagram 4

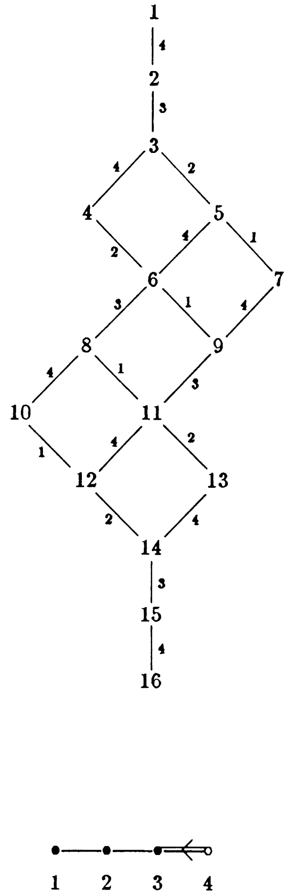


Diagram 5

6. C_n examples.

In this section we consider examples of type C_n. It is conceivable in general that $\mathcal{O}^\lambda$ depends only on W_λ up to category equivalence, so that the categories arising from B_n and C_n Lie algebras are equivalent. In any case, in the examples below, the computation of θ_α's and composition factors of generalized Verma modules depends only on ${}^SW_\lambda$, and from these computations the Conjecture will follow as in §4.7. Thus, each C_n example has a B_n-analogue.

Let R be a root system of type C_n with base $B = \{\alpha_1, \ldots, \alpha_n\}$, numbered in the obvious order with α_n the long root.

6.1 Let $S = \{\alpha_2, \ldots, \alpha_n\}$. Then ${}^SW$ is totally ordered of length $2n$ and the situation parallels that in §5.1. Each generalized Verma module besides $M_S({}^Sw_0 \cdot \lambda)$ has length two [**3**, 4.6], and the projective indecomposables besides $M_S(\lambda)$ are the images of simples under θ_i's.

6.2 Let $S = \{\alpha_1, \alpha_3, \ldots, \alpha_n\}$. The composition factors of generalized Verma modules are computed in [**2**]. This example is analogous to the A_n example of §5.2, and the Conjecture can be verified by following the same procedure. In particular, one finds that $M_S(\lambda)$ is uniserial of length 4 with socle $L({}^Sw_0 \cdot \lambda)$.

1	**2**	**3**	**4**
2	3 10	4 8 5	6
10	8	6	13
5	**6**	**7**	**8**
6 7	8 13 9	9	10 11
9	11	16	12
9	**10**	**11**	**12**
11 16	12	12 15 13	14
15		14	
13	**14**	**15**	**16**
14	15	16	

Diagram 6

6.3 Let $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$. The Bruhat poset has 2^n points and consists of two copies of the poset for the C_{n-1} case, glued together. We have only treated these cases for $n \leq 5$, and will discuss each in turn. The case $n = 2$ is covered by §6.1, and if $n = 3$, the poset ${}^S W$ coincides with the one in the example in §7.1 for D_4. The results are the same and we postpone discussion until then, except to note that $\nu = {}^S w_0 \cdot \lambda$ and ${}^S X' = \{s_3 s_2 s_1 s_3\}$.

Assume $n = 4$. The Bruhat poset is depicted in Diagram 5, and the procedure of §4.7 works smoothly.

PROPOSITION. *Let R be of type C_4 with $S = \{\alpha_1, \alpha_2, \alpha_3\}$, and let $z = s_4 s_3 s_2 s_4$.*
(i) *The socle of $M_S(\lambda)$ is $L(z s_3 s_4 \cdot \lambda)$.*
(ii) *z is special and is the unique element of ${}^S X'$.*
(iii) *$P_S(w \cdot \lambda)$ is self-dual for all $w \in {}^S X$.*

PROOF. The composition factors of generalized Verma modules can be calculated as in [**2**], and are given in Diagram 6.

The module $M_S(\lambda)$ has composition factors $L(1), L(2)$, and $L(10)$, and if $L(10)$ is not the socle, we obtain a contradiction by applying θ_4 to $M_S(1)$—the resulting module will have a copy of $L(10)$ in its top, whereas the top is actually $L(2)$. The element z is special by §4.5 and by §4.6 all generalized Verma modules $M_S(r)$ with $1 \leq r \leq 9$ or $r = 11$ have simple socle. But all the other generalized Verma modules have length one or two, involving only standard simples, and the socles are all $\gg z$. $\square$

Let us summarize the facts if $n = 5$. The composition factors of generalized Verma modules can be computed as usual, and $M_S(\lambda)$ is uniserial of length 4, the factors from top to bottom being $L(\lambda), L(s_5 \cdot \lambda), L(s_5 s_4 s_3 s_5 s_4 s_5 \cdot \lambda), L({}^S w_0 \cdot \lambda)$. The set ${}^S X'$ has one element, ${}^S w_0 s_5 s_4 s_3 s_5 s_4 s_2 \cdot \lambda$, and it is special.

REMARK. In the cases $n \leq 4$, the two hypotheses of Enright and Shelton discussed in the introduction are easily verified. For instance, with $n = 4$, we saw in the proof that the socles are simple. For $r \leq 9$ or $r = 11$, the modules $D_S(r)$ are simply $P_S(r)$, which are self-dual. For the remaining r, by definition and the known composition factors $D_S(r)$ has a Verma flag with two constituents, and simple top

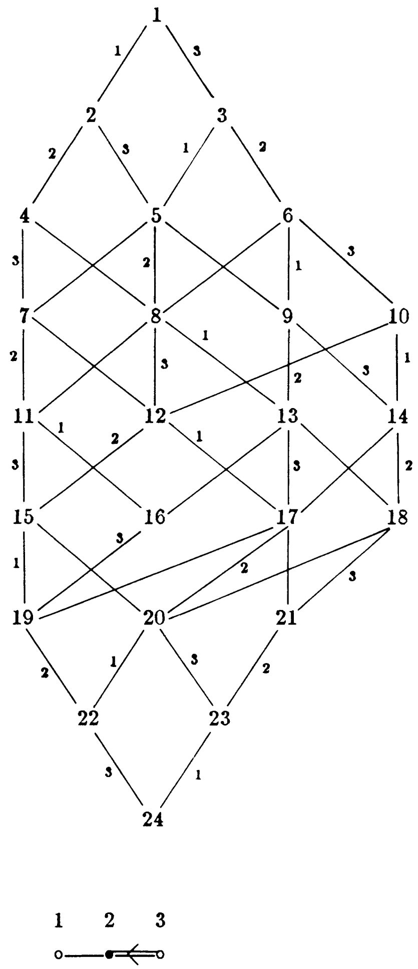


Diagram 7

$L(r^{\#})$ (see Diagram 6). It is easy to see that for suitable i the module $\theta_i L(r^{\#})$ has a Verma flag with the same constituents and same top, so the two coincide and $D_S(r)$ is self-dual.

If $n = 5$, one can still show that the generalized Verma modules have simple socle. As with $n = 4$, the only potentially troublesome cases are handled by the procedure of §4.7 in applying §4.6. The other generalized Verma modules can be shown to have simple socle by considering standard maps.

6.4 The Conjecture is true for all parabolics in C_3, the rank 2

parabolics being covered by the earlier subsections. For the remaining three cases, we will describe some of the facts, treating one case in detail. In all the cases, the composition factors of generalized Verma modules can be computed from the known information for C_3 Verma modules [**9**, 5.24].

(i) $S = \{\alpha_1\}$. In this case, $\nu = s_3 s_2 s_3 \cdot \lambda$ and ${}^S X' = \{s_3 s_2 s_3\}$, with $s_3 s_2 s_3$ special.

1	**2**	**3**	**4**	**5**	**6**
2 3	4 5	5 6 21	7 8	7 8 9 16 17	8 9 10 18
5 21	7 8 16 17	8 9 16 17 18	11 12 20	11 12 13 19 20	12 13 14 20
16 17	11 12 19 20	13 19 20	15	15 22	17 22
19	15	22			24

7	**8**	**9**	**10**	**11**	**12**
11 12	11 12 13 20	13 14	12 14	15 16	15 17
15	15 16 17 22	17 18 22	17	19	19 20 24
	19 24	20 21 24	24		22
		23			

13	**14**	**15**	**16**	**17**	**18**
16 17 18 22	17 18	19 20	19	19 20 21 24	20 21
19 20 21 24	20 21 24	22		22 23	23
23	23				

19	**20**	**21**	**22**	**23**	**24**
22	22 23	23	24	24	
	24				

Diagram 8

(ii) $S = \{\alpha_3\}$. In this case $\nu = {}^S w_0 \cdot \lambda$ and ${}^S X' = \{y, z\}$, with $y = {}^S w_0 s_2 s_1$ and $z = {}^S w_0 s_2 s_3$. Also ${}^S X = \{y, z, {}^S w_0 s_2, {}^S w_0\}$ and $\check{y} = s_2 s_1$, $\check{z} = s_2 s_3$, ${}^S \widecheck{w_0 s_2} = s_2$. The one feature of this example which distinguishes it from the previous ones is that there is an element z of ${}^S X'$ with $\check{z} \neq z\bar{z}$. This is presumably a common phenomenon.

(iii) $S = \{\alpha_2\}$. This is the most interesting of the C_3 examples, different from all the other examples in the paper in one respect. The poset ${}^S W$ is in Diagram 7 and the generalized Verma modules are in Diagram 8.

PROPOSITION. (i) *The socle of* $M_S(1)$ *is* $L(19)$.

(ii) *The set* ${}^S X' = \{15, 23\}$, *and these elements are special.*

(iii) ${}^S X = \{15, 19, 22, 24, 23\}$ *and* ${}^S Y = \{2, 1, 3, 6, 9\}$, *with these sets listed in parallel order with respect to* ${}^\vee$.

(iv) $P_S(r)$ *is self-dual for* $r \in {}^S X$.

REMARK. The new feature of this example is that ${}^S X'$ has two elements, 15 and 23, of different length, with $15 < 23$ in the Bruhat order (among others), but under the $\ll$ order, they are incomparable.

PROOF. (i) There are four nonstandard composition factors in $M_S(1)$, and we must sort out which is the lowest. By [**9**, 5.24], these four factors have multiplicity 1 or 2 in any Verma module. A consideration of standard maps into $M_S(5)$ shows that the factors $L(16), L(17)$, and $L(19)$ in $M_S(5)$ are not standard, so they cannot be in the kernel of the standard map $M_S(5) \to M_S(1)$ (if one is, its multiplicity in $M(1)$ must be ≥ 3). Since the image of $M_S(5)$ in $M_S(1)$ does not involve $L(21)$, it cannot be the socle. By applying θ_α to $M_S(1)$, we find that $L(16)$ or $L(17)$ cannot be the socle either—if $L(16)$ (or $L(17)$) is, then $L(11)$ (or $L(12)$) must be in the socle of $M_S(2)$. But $L(11)$ and $L(12)$ are standard factors of $M_S(2)$, lying above $L(15)$.

(ii)(iii) The specialness of 15 and 23 follows by §4.5, and so ${}^SX$ contains the indicated set. We can show no other simples are socular by examining standard maps. For instance, in $M_S(5)$ there are five nonstandard factors, and we already know $L(19)$ lies below $L(16)$ and $L(17)$ in the socle series by using the standard map to $M_S(1)$. We must also make sure $L(20)$ is not in the socle. In the standard map to $M_S(2)$, the simple $L(20)$ must be in the image (or else, as above, it has multiplicity 3 in $M(2)$). But the socle of $M_S(2)$ is $L(15)$, by §4.6, so $L(20)$ lies above the standard $L(15)$ in $M_S(5)$. Thus the socle of $M_S(5)$ is in $L(15) \oplus L(22)$, and it is easy to see that this is the socle. This is the hardest case. As usual, the rest of the Proposition follows by §4.4. □

7. D_n examples. In this section, R is a root system of type D_n, with base $B = \{\alpha_1, \ldots, \alpha_{n-3}, \eta, \beta, \gamma\}$, as in Diagram 9.

7.1 Let $S = \{\alpha_2, \ldots, \alpha_{n-3}, \eta, \beta, \gamma\}$. The determination of composition factors in this case is due to Borho and Jantzen [**3**, 4.6]. The poset ${}^SW$ and composition factors are listed in Diagrams 9 and 10.

PROPOSITION. (i) *The socle of* $M_S(1)$ *is* $L(2n)$.
(ii) *The element* $n+2$ *is special and* ${}^SX' = \{n+2\}$.
(iii) $P_S(r)$ *is self-dual for* $r \geq n+2$.

PROOF. This example may be viewed as a thinned-down version of the A_n example in §5.2, and the arguments given there carry over directly, becoming more transparent in the process. □

REMARKS. By §4.6, we find that $M_S(r)$ has simple socle $L(2n+1-r)$ for $r \leq n-1$. But for $r \geq n$, the simples in $M_S(r)$ are standard and $M_S(r)$ is uniserial of length 1 or 2. We can deduce, as in §6.3, that $D_S(r) = P_S(r)$ for $r \leq n-1$ and $D_S(r) = \theta_\alpha L(r^\#)$ for $r \geq n$, where α is a simple root depending on r. In particular, all $D_S(r)$'s are self-dual, as Enright and Shelton proved in [**5**].

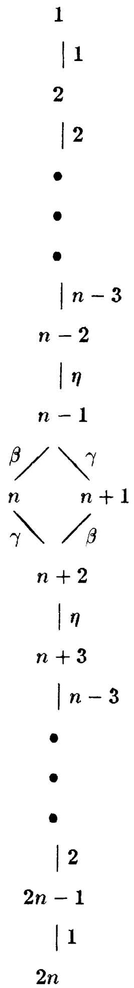
7.2 Let R be of type D_5 with $S = \{\alpha_1, \alpha_2, \eta, \beta\}$. The poset ${}^SW$ is identical to the poset in §6.3 for type C_4, and the results are parallel.

7.3 Let R be of type D_4 with $S = \{\alpha, \beta, \gamma\}$. The Bruhat poset is in Diagram 11, and the composition factors of generalized Verma modules are in Diagram 12.

These can be calculated as in [**2**], except for one tricky point, showing that $\theta_\gamma(17)$ involves $L(24)$ in its middle layer (and similarly for $\theta_\beta(18), \theta_\alpha(19)$). At this stage in the computation, one already knows the structure of $M_S(17)$, which has socle $L(23)$ and a factor $L(24)$ above it. Applying θ_γ to $M_S(17)$ and assuming $(\theta_\gamma L(17) : L(24)) = 0$, we find that $L(24)$ does not lie above $L(23)$ in $M_S(17)$, a contradiction. The rest of the calculation is straightforward.

PROPOSITION. (i) $M_S(1)$ *has socle* $L(16)$.
(ii) *The element* 12 *is special, and* ${}^SX' = \{12\}$.
(iii) $P_S(r)$ *is self-dual for* $r \in {}^SX$.

PROOF. Examining the composition factors of $M_S(1)$, we find by symmetry that the only candidates for socle are $L(2)$ and $L(16)$. If $L(2)$ is the socle, it cannot be the top of $\theta_\eta M_S(1)$, a contradiction.



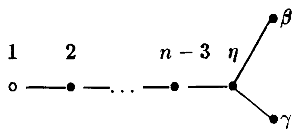


Diagram 9

(ii) By §§4.5 and 4.4, we find 12 is special and the set $\{r|r \gg 12\}$ is socular. By §4.6, the socle of $M_S(r)$ is simple for $r \leq 8$ and $r = 12, 16$. We need only examine the socle of $M_S(r)$ for $r \in \{9, 10, 11, 13, 14, 15\}$ to make sure no other simples are socular. The factors of $M_S(9)$ are all standard and $M_S(9)$ has socle $L(17)$. Similarly, $M_S(10)$ and $M_S(11)$ have socle $L(18)$ and $L(19)$. If $L(17)$ is in the socle of $M_S(13)$, it cannot be the top of $\theta_n M_S(13)$, a contradiction, so $M_S(13)$ is uniserial with socle $L(24)$. The same holds for $M_S(14)$ and $M_S(15)$. $\square$

$$\begin{array}{c} 1 \\ 2 \\ 2n \end{array}$$

$$\begin{array}{c} r \\ r+1 \quad 2n+2-r \\ 2n+1-r \end{array} \qquad (2 \le r \le n-2)$$

$$\begin{array}{c} n-1 \\ n \quad n+3 \quad n+1 \\ n+2 \end{array}$$

$$\begin{array}{c} n \\ n+2 \end{array}$$

$$\begin{array}{c} n+1 \\ n+2 \end{array}$$

$$\begin{array}{c} n+r \\ n+r+1 \end{array} \qquad (1 \le r \le n-1)$$

$$2n$$

Diagram 10

REMARKS. (1) The proof of (ii) shows that every generalized Verma module has simple socle. All $M_S(r)$ for $r \leq 16$ were explicitly treated. If $r = 17, 18, 19$, the socle is $L(23)$—the only nonstandard simple is $L(24)$ and it is not the socle, as mentioned before the Proposition. One sees this in applying θ_β to $M_S(20)$, etc. In particular, the modules $D_S(r)$ can be defined, and it turns out that $D_S(r)$ is *not* self-dual if $r = 9, 10, 11$. Explicitly $D_S(9)$ is a homomorphic image of $P_S(17)$, having a Verma flag with factors $M_S(9)$, $M_S(13)$, $M_S(16)$, $M_S(17)$. Hence, $D_S(9)$ has as a submodule an extension of $M_S(9)$ by $M_S(16)$. The socle of $M_S(16)$ is $L(23)$, and there is no nontrivial extension between $L(23)$ and any composition factor of $M_S(9)$. To see this, note that any factor $L(r)$ of $M_S(9)$ satisfies $r < 23$ in the Bruhat order, and if $L(r)$ extends $L(23)$ nontrivially, the extension E is a highest weight module. Hence E is a homomorphic image of $M_S(r)$, and inspection of the four cases shows this does not happen. Therefore $L(17) \oplus L(23)$ is in the socle of any extension of $M_S(9)$ by $M_S(16)$, and in the socle of $D_S(9)$.

(2) In the example of §5.3 with $\mathfrak{g}$ of type A_4 and $\mathfrak{g}_S$ of type A_2, we omitted the details, but one finds that all generalized Verma modules have simple socle. (In fact, only four of the 20 generalized Verma modules have nonstandard composition factors, and they have simple socle by §4.6.) Thus, the modules $D_S(w \cdot \lambda)$ are defined here too, and by an argument similar to that above, one finds that $D_S(w \cdot \lambda)$ is not self-dual for $w = s_1$ and $w = s_1 s_2 s_1$.

8. Exceptional examples.

8.1 Let R be the E_6 root system with base $B = \{\alpha_1, \ldots, \alpha_6\}$ numbered as in Diagram 13. Let $B \backslash S = \{\alpha_1\}$. The composition factors of generalized Verma

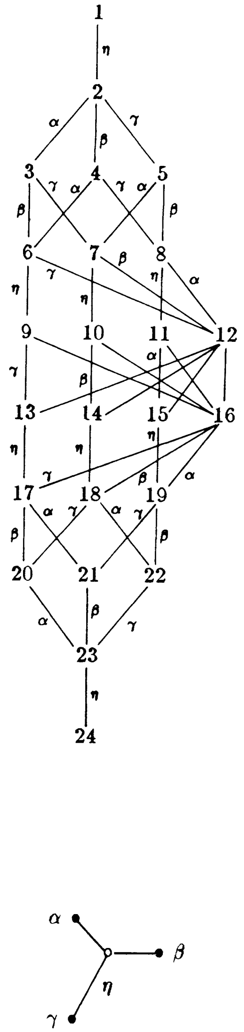


Diagram 11

modules were computed by Borho and Jantzen [**3**, 4.6], and the algorithm in [**2**] works as well; see Diagram 14.

PROPOSITION. (i) *The socle of* $M_S(1)$ *is* $L(15)$.
(ii) ${}^SX' = \{8\}$ *and* 8 *is special.*
(iii) $P_S(r)$ *is self-dual for* $r \in {}^SX$.
(iv) *Every generalized Verma module has simple socle.*
(v) $D_S(r)$ *is self-dual for all* r.

PROOF. The usual arguments work. If $L(15)$ is not the socle of $M_S(1)$, applying θ_2 to $M_S(1)$ yields a contradiction. The specialness of 8 is clear, and the self-duality of $P_S(r)$ for $r \geq 8$ follows. This in turn implies that $M_S(r)$ has simple socle for

1	**2**		**3**	**4**	**5**	**6**
2	3 4 5	9 10 11	6 7 15	6 8 14	7 8 13	9 12
9 10 11	6 7 8	13 14 15 16	12	12	12	13 16 20 21
16	12		20	21	22	17
7	**8**	**9**	**10**	**11**	**12**	
10 12	11 12	13 16	14 16	15 16	13 14 15 16	20 21 22
14 16 20 22	15 16 21 22	17	18	19	17 18 19	
18	19				24	
13	**14**	**15**	**16**	**17**	**18**	
17	18	19	17 18 19	20 24 21	20 24 22	
24	24	24	20 21 22 24	23	23	
			23			
19	**20**	**21**	**22**	**23**	**24**	
21 24 22	23	23	23	24		
23						

Diagram 12

$1 \leq r \leq 19$ and $r = 21$. But all other generalized Verma modules have length 1 or 2, so their socles are simple.

It follows exactly as in §§6.3 and 7.1 that $D_S(r)$ is self-dual for all r. Specifically, for $r \leq 19$ and $r = 21$, we have $D_S(r) = P_S(r^{\#})$. For the remaining r, we have $D_S(r) = \theta_i L(r^{\#})$ for a suitable i, as we see by comparing composition factors of the two modules, each of which have simple top $L(r^{\#})$. $\square$

8.2 Let R be of type E_7, with R_S of type E_6. The composition factors of generalized Verma modules have been computed by D. Collingwood in [**18**], and we will adopt his notation, in which the weights are numbered from 0 to 55, with 55 dominant.

PROPOSITION. (i) *The socle of* $M_S(55)$ *is* $L(0)$.
(ii) ${}^SX' = \{23\}$ *and* 23 *is special.*
(iii) $P_S(r)$ *is self-dual for all* $r \in {}^SX$.
(iv) *Every generalized Verma module has simple socle.*
(v) $D_S(r)$ *is self-dual for all* r.

PROOF. (i)–(iv) The proof of (i) is of the usual sort, and the specialness of 23 follows immediately. It only remains to check that the socle of $M_S(r)$ for $r \notin \{\check{s} | s \leq\leq 23\}$ is simple with highest weight $\leq\leq 23$. In fact, the structure of $M_S(r)$ for these r's can be computed directly while one computes the composition factors, and we obtain (i)–(iv).

(v) For $t \in {}^SZ$, we have $D_S(t) = P_S(t)$, so we may assume $t \notin {}^SZ$. For $0 \leq t \leq 6$ and $t = 8$ it is clear that $D_S(t) = \theta_\alpha L(i)$ for suitable α and i, and for the remaining t one can prove inductively that $D_S(t) = \theta_\beta D_S(j)$ for suitable β and j with $D_S(j)$ already known to be self-dual. For instance, suppose $D_S(12)$ is self-dual. The module $\theta_e D_S(12)$ is also self-dual, with a Verma flag whose constituents from top to bottom are $M_S(r)$ for $r = 5, 7, 12, 15$. A summand with $M_S(15)$ as submodule must be self-dual, since $L(15)$ has multiplicity 1 in $\theta_e D_S(12)$. The socle of $M_S(15)$ is $L(5)$, so the summand under consideration also involves $M_S(5)$, from which we see that $\theta_e D_S(12)$ is indecomposable. The module $\theta_e M_S(7)$ is a homomorphic image with simple top $L(5)$, from which we deduce that $L(7)$ is not in the top, which

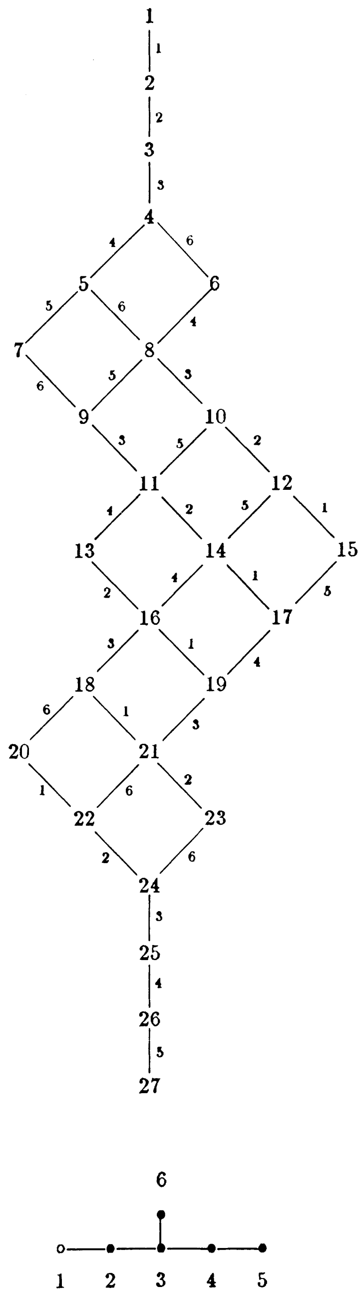


Diagram 13

1	**2**	**3**	**4**	**5**	**6**	**7**
2	3 15	4 12	5 10 6	7 8	8	9
15	12	10	8	9	13	20
8	**9**	**10**	**11**	**12**	**13**	**14**
9 13 10	11 20	11 12	13 18 14	14 15	16	16 17
11	18	14	16	17	23	19
15	**16**	**17**	**18**	**19**	**20**	**21**
17	18 23 19	19 27	20 21	21 26	22	22 25 23
27	21	26	22	25		24
22	**23**	**24**	**25**	**26**	**27**	
24	24	25	26	27		

Diagram 14

must therefore be simple. We may conclude that $\theta_e D_S(12)$ is a homomorphic image of $P_S(5)$ with the same Verma flag constituents as $D_S(15)$, implying that $D_S(15) = \theta_e D_S(12)$. A similar argument suffices for the other cases. $\square$

8.3 Let R be of type F_4, with $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ as in Diagram 15 and $S = \{\alpha_1, \alpha_2, \alpha_3\}$. The poset ${}^SW$ is in Diagram 15, and the composition factors of generalized Verma modules are computed in [**2**].

PROPOSITION. (i) *The socle of* $M_S(1)$ *is* $L(24)$.
(ii) ${}^SX' = \{15\}$ *and* 15 *is special.*
(iii) $P_S(r)$ *is self-dual for* $r \in {}^SX$.

PROOF. (i) The composition factors of $M_S(1)$ are $L(r)$ for $r = 1, 2, 13, 24$, each with multiplicity one. If $L(13)$ is the socle, applying $\theta_2\theta_1\theta_3\theta_2\theta_3\theta_4$ to $M_S(1)$ leads to the conclusion that $L(14)$ is in the socle of $M_S(9)$, which an examination of standard composition factors of $M_S(9)$ shows to be false.

(ii)(iii) The specialness of 15 follows easily, using §4.5. We need only check that the only socular weights are those $\gg 15$, and by §4.6 and the list of composition factors, the only generalized Verma modules to examine are $M_S(r)$ for $r = 8, 10 \to 14, 16$. Examining standard composition factors disposes of all cases but $M_S(8)$ and $M_S(10)$. The composition factors of $M_S(8)$ are $L(8), L(10)$, and $L(20)$, and if $L(20)$ is not the socle, then $\theta_1 M_S(8)$ has $L(20)$ in its top, a contradiction. Similarly, $L(12)$ cannot be in the socle of $M_S(10)$, or else $L(17)$ is in the top of $\theta_2 M_S(10)$. $\square$

REMARK. The module $M_S(12)$ has socle $L(15) \oplus L(19)$, as one can see by examining $\theta_3 M_S(14)$, once one checks that $M_S(14)$ has socle $L(17)$ (the one nonstandard factor $L(19)$ of $M_S(14)$ lies above $L(17)$ in the socle series). In particular, the module $D_S(12)$ is not defined.

9. Miscellaneous examples and nonregular weights.

9.1 We have only considered regular weights $\lambda \in P_S^+$ in our examples, but the Conjecture makes sense for nonregular weights as well. Moreover, for μ nonregular in P_S^+, the category $\mathcal{O}_S^\mu$ is likely to be less complicated than $\mathcal{O}_S^\lambda$, and information in $\mathcal{O}_S^\mu$ can be translated to $\mathcal{O}_S^\lambda$ via T_μ^λ. In fact, this provides an easy proof of the Conjecture for integral weights if $S = \varnothing$, the proof of Humphreys mentioned

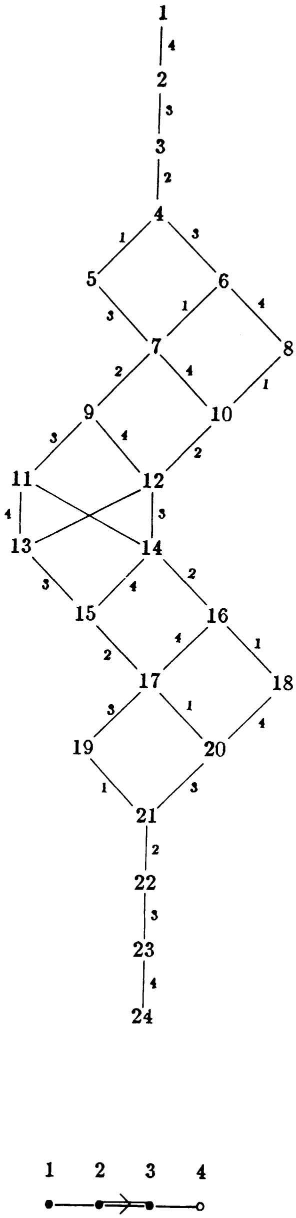


Diagram 15

in the introduction: Assume λ is dominant integral. The weight $-\rho$ has orbit $W \cdot \{-\rho\} = \{-\rho\}$, so $\mathcal{O}^{-\rho}$ has one nonzero module, $L(-\rho)$. Applying $T_{-\rho}^{\lambda}$ to $L(-\rho)$ produces a self-dual projective P in $\mathcal{O}^{\lambda}$. But the only socular simple in $\mathcal{O}^{\lambda}$ is $L(w_0 \cdot \lambda)$, so that the top of P must be a direct sum of copies of $L(w_0 \cdot \lambda)$ and P is a direct sum of copies of $P(w_0 \cdot \lambda)$.

In a similar manner, we will produce self-dual projectives in $\mathcal{O}_S^\lambda$ for certain choices of S by translating from $\mathcal{O}_S^\mu$ for μ chosen to be as degenerate as possible in P_S^+. We cannot expect success from this approach in general, since μ must be W_S-regular. Thus, the larger S is, the more regular μ is and the more complicated the category $\mathcal{O}_S^\mu$ becomes, so that we may not be able to find any self-dual projectives in $\mathcal{O}_S^\mu$. If $\mathcal{O}_S^\mu$ *does* have a self-dual projective, so does $\mathcal{O}_S^\lambda$, and the Conjecture will follow from §9.2, in which we prove that the self-duality of one projective in $\mathcal{O}_S^\lambda$ implies the self-duality of all $P_S(w \cdot \lambda)$ for $w \in {}^S X_\lambda$.

9.2 Let λ be a dominant, regular weight in P_S^+ for some choice of S. We saw in §4.3 that the set of socular elements in ${}^S W_\lambda$ is the right cell of W_λ containing ${}^S w_0$. An important consequence of the Kazhdan-Lusztig Conjecture is that the preorder $\leq_R$ on W_λ, in terms of which right cells are defined, is generated by the following three cases, where w is an element of W_λ.

(i) $w \leq_R w$.

(ii) If $\alpha \in B_\lambda$ and $w < ws_\alpha$, then $w \leq_R ws_\alpha$.

(iii) If $y \in W_\lambda$ and $\alpha \in B_\lambda$ with $ys_\alpha < y < w < ws_\alpha$, and if $\mu(w, y) \neq 0$, then $w \leq_R y$.

The function μ is the one arising in the definition of Kazhdan-Lusztig polynomials. We refer to [**10**, Chapter 16] for additional details. Recall that one defines $y \sim_R w$ if $y \leq_R w$ and $w \leq_R y$, and a right cell is an equivalence class of W_λ under $\sim_R$.

THEOREM. *Let λ be a dominant, regular weight in P_S^+. If $P_S(w \cdot \lambda)$ is self-dual and y is an element of ${}^S W_\lambda$ such that $w \leq_R y$, then $P_S(y \cdot \lambda)$ is self-dual.*

PROOF. It suffices to prove the Theorem for y arising as in cases (ii) and (iii) above, and case (ii) is included in the Theorem in §4.4. Thus we may assume y is chosen as in case (iii). By hypothesis, θ_α annihilates $L(w \cdot \lambda)$, but not $L(y \cdot \lambda)$. Thus, if $L(y \cdot \lambda)$ is in the top of the radical of $P_S(w \cdot \lambda)$, then $L(y \cdot \lambda)$ is in the top of $\theta_\alpha P_S(w \cdot \lambda)$. This implies that $P_S(y \cdot \lambda)$ is a summand, and by §4.2, $P_S(y \cdot \lambda)$ is self-dual.

Therefore, it suffices to prove that $\operatorname{Ext}^1(L(y \cdot \lambda), L(w \cdot \lambda)) \neq 0$. It was proved in [**8**, **7.7**], and also in [**17**], that if Jantzen's Conjecture is satisfied in $\mathcal{O}^\lambda$, then $\mu(w, y) = \dim \operatorname{Ext}^1(L(y \cdot \lambda), L(w \cdot \lambda))$. (As noted in [**8**], this is essentially in [**6**, 3.16].) Jantzen's Conjecture has been proved by J. Bernstein [**14**], and the Theorem follows. □

COROLLARY. *Let λ be a dominant, regular weight in P_S^+. If $\mathcal{O}_S^\lambda$ contains a self-dual, projective module, then $P_S(y \cdot \lambda)$ is self-dual for all $y \in {}^S X_\lambda$.*

PROOF. By §4.2, some indecomposable projective $P_S(w \cdot \lambda)$ is self-dual, with w necessarily in ${}^S X_\lambda$. But ${}^S X_\lambda$ is a right cell (§4.3), so that for any $y \in {}^S X_\lambda$ we have $w \leq_R y$, and the Theorem applies. □

REMARK. In §§5–8 we have relied on the Theorem in §4.4, which allows one to deduce the self-duality of $P_S(y \cdot \lambda)$ from $P_S(w \cdot \lambda)$ only for $y \geq\geq w$. Use of the Theorem or Corollary makes it possible to shorten a little of the work of those sections, if one only wants to verify the Conjecture, since one needs only to find one special element of ${}^S X_\lambda$. We will use the Theorem in the rest of this section.

9.3 Let R be of type A_n and let $S = \{\alpha_1\}$, with $T = \{\alpha_2, \ldots, \alpha_n\}$, using the notation of §5. Recall that w_T denotes the longest element of W_T.

PROPOSITION. *Let λ be a regular, dominant integral weight. The socle of $M_S(\lambda)$ is $L(w_T\cdot\lambda)$ and $P_S(w_T\cdot\lambda)$ is self-dual. Hence, $P_S(w\cdot\lambda)$ is self-dual for all $w\in {}^SX$.*

PROOF. Let μ be a dominant integral weight with $(\mu+\rho,\check{\alpha})=0$ for $\alpha\in T$. The set $W\cdot\mu$ consists of the weights $\{s_rs_{r-1}\cdots s_1\cdot\mu|0\le r\le n\}$ and s_1 fixes all these weights except μ and $s_1\cdot\mu$. Thus $W\cdot\mu\cap P_S^+=\{\mu\}$ and the only nonzero module of $\mathcal{O}_S^\mu$ is $L(\mu)$, which equals $M_S(\mu)$ and $P_S(\mu)$.

The module $T_\mu^\lambda L(\mu)$ is a self-dual projective of $\mathcal{O}_S^\lambda$, and the constituents of its Verma flag can be computed via [**9**, 2.17]. The module $T_\mu^\lambda M(\mu)$ has a Verma flag with constituents $\{M(w\cdot\lambda)|w\in W_T\}$ and $T_\mu^\lambda M(s_1\cdot\mu)$ has a flag involving $\{M(s_1w\cdot\lambda)|w\in W_T\}$. Since $M_S(\mu)=M(\mu)/M(s_1\cdot\mu)$ and T_μ^λ is exact, we find that the Verma flag of $T_\mu^\lambda L(\mu)$ has constituents $\{M_S(w\cdot\lambda)|w\in W_T\}$. In particular, $w_T\cdot\lambda$ is the lowest weight occurring, implying that $P_S(w_T\cdot\lambda)$ is a summand of $T_\mu^\lambda L(\mu)$. On the other hand, by [**9**, 2.16], for any $w\in W_T$ we have $(M(w\cdot\lambda):L(w_T\cdot\lambda))=1$, and since $s_1w\not<w_T$, we obtain $(M(s_1w\cdot\lambda):L(w_T\cdot\lambda))=0$, and $(M_S(w\cdot\lambda):L(w_T\cdot\lambda))=1$. Thus, by BGG reciprocity, $P_S(w_T\cdot\lambda)$ has in its Verma flag all the constituents of $T_\mu^\lambda L(\mu)$, implying that the two modules coincide. Lastly, $M_S(\lambda)$ is a submodule of $P_S(w_T\cdot\lambda)$, which has simple socle by self-duality, so $L(w_T\cdot\lambda)$ is the socle. □

9.4 Let R be of type A_n with $S=\{\alpha_1,\alpha_2\}$ and $T=B\backslash S$.

PROPOSITION. *Let λ be a regular, dominant integral weight. The socle of $M_S(\lambda)$ is $L(w_T\cdot\lambda)$ and $P_S(w_T\cdot\lambda)$ is self-dual. Hence, $P_S(w\cdot\lambda)$ is self-dual for all $w\in {}^SX$.*

PROOF. Let μ be a dominant integral weight with $(\mu+\rho,\check{\alpha})=0$ for $\alpha\in T$. The set $W\cdot\mu$ has Bruhat order essentially that in Diagram 4, provided we understand that weights η and ς with $\eta>\varsigma$ are connected by an edge i if $\varsigma=s_i\cdot\eta$. An induction on n yields the result that $W\cdot\mu\cap P_S^+=\{\mu\}$, so that $\mathcal{O}_S^\mu$ has only $L(\mu)$ as a nonzero module. The Proposition follows by the same argument used in §9.3. □

9.5 Let R be of type A_n with $S=\{\alpha_1,\alpha_n\}$ and $T=B\backslash S$.

PROPOSITION. *Let λ be a regular, dominant integral weight. The socle of $M_S(\lambda)$ is $L({}^Sw_0\cdot\lambda)$ and $P_S({}^Sw_0\cdot\lambda)$ is self-dual. Hence, $P_S(w\cdot\lambda)$ is self-dual for all $w\in {}^SX$.*

PROOF. Let μ be a dominant integral weight with $(\mu+\rho,\check{\alpha})=0$ for $\alpha\in T$. The Bruhat order of $W\cdot\mu$ is essentially given in Diagram 2, with the same proviso as in §9.4. Inspection shows that $W\cdot\mu\cap P_S^+=\{\mu,\eta\}$ for $\eta=s_2s_3\cdots s_{n-1}\cdots s_3s_2s_1s_n\cdot\mu=s_1s_nw_0\cdot\mu$. It is not hard to calculate the composition factors of Verma modules in $\mathcal{O}^\mu$ from [**9**, Chapter 5] (see for instance [**8**, 11.1]), from which one finds that $M_S(\mu)$ is uniserial of length two with socle $L(\eta)$, and $M_S(\eta)=L(\eta)$. Thus $P_S(\eta)$ is an extension of $M_S(\mu)$ by $L(\eta)$, and is uniserial of length three, while $P_S(\mu)=M_S(\mu)$. In particular, $P_S(\eta)$ is self-dual, and $T_\mu^\lambda P_S(\eta)$ is a self-dual, projective in $\mathcal{O}_S^\lambda$.

As in §9.3, one obtains a Verma flag for $T_\mu^\lambda P_S(\eta)$ with constituents

$$\{M_S(w\cdot\lambda)|w\in W_T\}\cup\{M_S(s_1s_nww_0\cdot\lambda|w\in W_T\},$$

using [**9**, 2.17]. But as w runs over all the elements of W_T, so does $w_0ww_0^{-1}$, since w_0 stabilizes T, so the second set consists of modules $M_S(s_1s_nw_0w\cdot\lambda)$ for $w\in W_T$. Since $s_1s_nw_0={}^Sw_0$, we may view the second set as $\{M_S({}^Sw_0w\cdot\lambda)|w\in W_T\}$.

In particular, $M_S({}^S w_0 \cdot \lambda)$ is the lowest generalized Verma module occurring, so $P_S({}^S w_0 \cdot \lambda)$ is a summand of $T_\mu^\lambda P_S(\eta)$, and all the modules of the form $M_S({}^S w_0 w \cdot \lambda)$ with $w \in W_T$ occur in the Verma flag of $P_S({}^S w_0 \cdot \lambda)$. If there is another summand P, all the constituents of its Verma flag must be of the form $M_S(w \cdot \lambda)$ with $w \in W_T$. Applying T_λ^μ to P, we obtain a self-dual projective in $\mathcal{O}_S^\mu$ whose Verma flag involves only $M_S(\mu)$, implying that $M_S(\mu)$ is self-dual, a contradiction. Therefore $T_\mu^\lambda P_S(\eta) = P_S({}^S w_0 \cdot \lambda)$ and $M_S(\lambda)$ is a submodule, proving the Proposition. □

REMARK. If R is of type D_n and $B \backslash S = \{\alpha_1\}$ in the notation of §7, the statement of the Proposition carries over with essentially the same argument.

9.6 In every example considered so far, any $(M_S(\eta) : L(\varsigma))$ which has been computed is 0 or 1. In particular, every self-dual projective considered has a given generalized Verma module in its Verma flag at most once. The resulting picture may be deceptive. Let us note in closing that a socular simple may have multiplicity > 1 in a generalized Verma module; in fact, the multiplicity can be > 1 in the socle of a generalized Verma module.

Let R be a root system of type D_4 with $S = \{\eta\}$, in the notation of §7. Let μ be a dominant weight in P_S^+ with $(\mu + \rho, \check{\delta}) = 0$ for $\delta \in B \backslash S$. The Bruhat order on $W \cdot \mu$ is depicted in Diagram 11, under the proviso of §9.4, and $W \cdot \mu \cap P_S^+$ consists of the weights numbered $1; 6, 7, 8; 12; 13, 14, 15; 23$. With the help of [**9**, 5.16] and direct calculation, one can determine the structure of the generalized Verma modules. In particular, $M_S(12)$ has socle $L(23) \oplus L(23)$, and $L(23)$ is the only socular simple. One would expect $P_S(23)$ to be self-dual, and its structure is complicated in a fundamental way, with two copies of $M_S(12)$ in any Verma flag. It turns out that a likely guess for the socle series of $P_S(23)$ has the property that its layers coincide with the layers of the upside-down series, suggesting self-duality.

Addendum (December, 1984).

A.1. The conjecture on self-duality can be easily proved along the lines of §4, the key point being an observation of Devra Garfinkle. The proof can be done in three steps. Recall, for a fixed subset S of B, that there is a maximum possible value for the Gelfand-Kirillov dimension of modules in $\mathcal{O}_S$, assumed by the generalized Verma modules [**10**, 8.6 and 15.3]. The first step is to treat $\mathcal{O}_S^\lambda$ for λ a dominant, regular weight in P_S^+ which vanishes on $\mathfrak{h}_S$. Then $M_S(\lambda)$ is critical with respect to Gelfand-Kirillov dimension [**10**, 15.5], which means that the simple socle $L(\nu)$ is the only composition factor of $M_S(\lambda)$ with maximal dimension. Given $w \in {}^S W_\lambda$, let θ be a composition of functors of the form θ_α, with $\alpha \in B_\lambda$, for which $P_S(w \cdot \lambda)$ is a summand of $\theta M_S(\lambda)$ (see 2.2(vi)). If $w \in {}^S X_\lambda$, then $L(w \cdot \lambda)$ has maximal dimension (4.3), and it immediately follows that $(\theta M_S(\lambda) \colon L(w \cdot \lambda)) = (\theta L(\nu) \colon L(w \cdot \lambda))$, since $d(\theta(M_S(\lambda)/L(\nu))) \le d(M_S(\lambda)/L(\nu)) < d(L(w \cdot \lambda))$. But the equality of multiplicities is precisely the statement that w is special, and the self-duality of $P_S(w \cdot \lambda)$ follows by 4.4. The observation that one can use the criticality of $M_S(\lambda)$ to prove specialness is due to D. Garfinkle.

For λ an arbitrary dominant, regular weight in P_S^+, one can find a weight λ' as above for which $\mathcal{O}_S^\lambda$ and $\mathcal{O}_S^{\lambda'}$ are equivalent categories, as in the argument of 4.1, so the conjecture extends to all regular weights. Let μ be a dominant weight and let w be an element of W_μ for which $L(w \cdot \mu)$ is socular in $\mathcal{O}_S$. Note that μ itself need not lie in P_S^+, since $(\mu + \rho, \check{\alpha})$ may be 0 for some α in S, but in any case

$S \subset R_\mu$. Thus there is a dominant regular λ in P_S^+ with $\lambda - \mu$ integral and μ in the lower closure of the facette of λ. Changing w if necessary, we may assume that $T_\lambda^\mu L(w \cdot \lambda) = L(w \cdot \mu)$ [**9**, 2.11]. Let y be an element of W_μ such that $L(w \cdot \mu)$ is a summand of the socle of $M_S(y \cdot \mu)$. Then $T_\mu^\lambda M_S(y \cdot \mu)$ has a Verma flag with $T_\mu^\lambda L(w \cdot \mu)$ as a submodule. But the adjointness of T_μ^λ and T_λ^μ implies that $L(w \cdot \lambda)$ is in the socle of $T_\mu^\lambda L(w \cdot \mu)$, so $L(w \cdot \lambda)$ is socular. Therefore, the previous argument yields the self-duality of $P_S(w \cdot \lambda)$, and since $P_S(w \cdot \mu)$ is a summand of $T_\lambda^\mu P_S(w \cdot \lambda)$, we obtain the self-duality of $P_S(w \cdot \mu)$.

A.2. It is easy to extend to $\mathcal{O}_S^\mu$, for μ nonregular, the conclusion of 4.3 that the socular simple modules are precisely those of maximal Gelfand-Kirillov dimension. If $L(w \cdot \mu)$ is socular, then the argument at the end of A.1 shows, changing w if necessary, that $L(w \cdot \mu) = T_\lambda^\mu L(w \cdot \lambda)$ for a socular simple $L(w \cdot \lambda)$. By [**9**, 3.4], we have $d(L(w \cdot \mu)) = d(L(w \cdot \lambda))$, yielding one direction. Conversely, given $L(w \cdot \mu)$ of maximal dimension and $L(w \cdot \lambda)$ as before, the equality of dimensions implies that $L(w \cdot \lambda)$ is socular. Therefore $P_S(w \cdot \lambda)$ is self-dual, as is $P_S(w \cdot \mu)$, implying that $L(w \cdot \mu)$ is socular.

A.3. As noted at the beginning of 9.1, the conjecture has an easy proof in the case of the category $\mathcal{O}^\lambda$, if λ is integral. Theorem 9.2 makes it feasible to try to extend this approach to $\mathcal{O}_S^\lambda$, at least for integral λ, and it is more effective than the results of §9 suggest. The reason is touched on in A.1—one can have a dominant weight μ not in P_S^+ and some $w \in W_\mu$ with $w \cdot \mu \in P_S^+$ (in §9, the weight μ was restricted unnecessarily to lie in P_S^+). If, in such a case, $W_\mu \cdot \mu \cap P_S^+ = \{w \cdot \mu\}$, then $P_S(w \cdot \mu) = L(w \cdot \mu)$ and translation yields a large family of self-dual projectives. For instance, as Brad Shelton observed, if R is of type A_n and $S \subset B$, there is always a dominant, integral (nonregular) weight μ with $|W \cdot \mu \cap P_S^+| = 1$, from which the conjecture follows for all $\mathcal{O}_S^\lambda$ with R of type A_n and λ integral. Shelton has treated several other cases as well from this point of view.

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